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Reserve prices in all-pay auctions with complete information

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We introduce reserve prices in the literature concerning all-pay auctions with complete information, and reconsider the case for the so-called Exclusion Principle (namely, the fact that the seller may find it in her best interest to exclude the bidders with the largest willingness to pay for the prize). We show that a version of it extends to our setting. However, we also show that the Exclusion Principle: a) does not apply if the reserve price is large enough; 2) does not extend if the seller regards bidders' valuations as identically independently distributed according to a monotonic hazard rate. Preliminary results for the case of independent ex-ante asymmetric bidders suggest that the case for it in settings with positive reserve prices is actually tenuous.

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1. Introduction

Auction models are prototypes of competitive settings, and they are used in several branches of the economic literature. In particular, the so-called (first-price) all-pay auction is used (among others) by Hillman and Riley (1989), Baye *et alii* (1993) and Che and Gale (1998) to model the lobbying process. This type of auction fits the lobbying game well, since a lobbyist's contribution is not typically returned if his efforts are unsuccessful,[1] and indeed this literature has elaborated a number of interesting results. In particular, Hillman and Riley (1989) prove that, if there is some asymmetry among bidders/lobbyists, the politically contestable rent is not totally dissipated even in the case of a large number of potential contenders. In addition, Baye *et alii* (1993) show that a seller/politician wishing to maximize her revenue may find it in her best interest to exclude certain lobbyists from the "finalist" short list (the so-called "Exclusion Principle"), particularly those lobbyists valuing the political prize most (in order to raise incentives to spend for the likely losers). Che and Gale (1998) show a somehow related result: namely, the imposition of an exogenous cap on individual lobbying contributions may have the adverse effect of increasing total expenditure (by increasing competition among lobbyists).

It has to be stressed that the quoted literature refers to the case of *complete* information (according to standard terminology: see e.g. Mas-Colell *et alii*, 1995: section 23, Appendix B): this means that, at the time of bidding, any detail of the setting is common knowledge to all the bidders, including others' evaluations of the prize. In particular, the working of the Exclusion Principle also requires that bidders' evaluations are known to the seller (at the time exclusion is decided), a rather unusual assumption in auction theory (in fact, a general rationale for using an auction mechanism is exactly the fact that how much bidders value the prize to be allocated among them is their private knowledge). In addition, it has been noticed by Gale and Stegeman (1994) that the Exclusion Principle depends on the assumption that the politician must award the prize (i.e., it cannot withhold it nor use take-it-or-leave-it offers).[2] This assumption can be justified in the lobbying setting if the politician is unable to refuse credibly to allocate the political rent: e.g., Baye *et alii* (1993) refer to the choice of a city to host the Olympic Games.

In this paper we discuss the case in which the seller can possibly use a different exclusion tool, namely a reserve price (a common mechanism in auction theory),[3] which does not require the seller to know bidders' evaluations. After characterizing the equilibrium of the all-pay auction with an exogenously given reservation price, we show that the seller would indeed prefer a strictly

[1] This feature is also shared by other economic and social games, such as patent races and sports.

[2] Gale and Stegeman (1994) discuss a mechanism in which the seller need not to award the prize to the highest bidder, and prove that it delivers to the seller an higher revenue.

[3] In a lobbying game a positive "reserve price" could perhaps be interpreted as the politician's ability to postpone (possibly *sine die*) the final decision.

positive reserve price, which also increases the overall outcome efficiency (it might decrease the efficacy of the lobbying process through higher rent dissipation). We show that a generalized version of the Exclusion Principle holds: namely, that a seller lacking the full-fledged bargaining ability to make a take-it-or-leave-it offer to the highest-evaluation bidder (the intuitive optimal mechanism for the seller) could still find optimal to exclude some bidders from her short list even when using a positive reserve price. However, the case for the Exclusion principle becomes weaker in such a setting, since it cannot apply if the reserve price is high enough.

We then discuss a setting in which, in an all-pay auction with complete information among the bidders, the seller is not fully informed while setting her reserve price and/or considering some exclusion. Namely, we consider the case the seller ex ante regards bidders' "ad-interim" valuations as unknown realizations of random variables.[4] In such a setting Menicucci (2006) strikingly shows that, even if the seller regards the bidders' private valuations as identically and independently distributed (iid), for some information structures excluding all but two bidders (randomly selected) increases the seller's expected revenue (yet another version of the Exclusion Principle). We characterize the optimal reserve price in such a setting and extend the result in Bertoletti (2008): namely, while using a positive reserve price, the seller wishes no exclusion if she regards bidders' valuations as iid distributed according to a monotonic hazard rate (a feature of many common distributions). Preliminary results for the case of independent but ex-ante asymmetric valuations seem to suggest that the case for the Exclusion Principle in settings with positive reserve prices is indeed tenuous.

2. The all-pay auction with complete information and a reserve price

Consider the following setting: n (risk-neutral) agents (the "buyers") bid for a prize (there is no resale possibility). Bidder i's (private) valuation of the prize is v_i ($i = 1, \dots, n$), and we order the bidders in such a way that $v_1 > v_2 > \dots > v_{n-1} > v_n > 0$.[5] The "rules" of the auction can include a reserve (minimum) price $p_r \geq 0$, i.e., a price below which the prize is not assigned. In particular, let us indicate with b_i the bid of agent i. In an (first-price) all-pay auction, bidder i receives the prize if $b_i > \mathrm{Max}\{b_{j \neq i}\}$ and $b_i \geq p_r$, and in that case his payoff is $v_i - b_i$, whereas his payoff is $- b_i$ if he loses (ties are broken randomly). Assuming $p_r = 0$, Hillman and Riley (1989), and Baye *et alii* (1993) and (1996) show that in the unique Nash equilibrium agent 1 uses the uniform distribution $F_1(b_1) = b_1/v_2$

[4] Actually, this is the standard assumption in a "complete information" setting: again see Mas-Colell *et alii* (1995).

[5] The possibility of ties in the valuations is ignored here. This can be justified by assuming that the v_i are ex-ante continuously independently distributed, so that case has *a priori* a zero probability (in an all-pay auction ties may imply the existence of multiple Nash equilibria which are not necessarily revenue equivalent: see Baye *et alii*, 1996 and footnote 9 below).

on the support $[0,v_2]$, while agent 2 uses $F_2(b_2) = 1 - v_2/v_1 + b_2/v_1$ on the same support (note that this amounts to the fact that agent 2 randomises between $b_2 = 0$ and the uniform distribution on $[0,v_2]$ with probabilities respectively $1 - v_2/v_1$ and v_2/v_1). Agents $j = 3, \ldots, n$ bid $b_j = 0$ with probability 1. The prize is then given to agent 1 with probability $1 - v_2/(2v_1) > ½$ and to agent 2 with probability $v_2/(2v_1) < ½$ (note that in the latter event the result is not ex-post efficient, and thus it would not be stable in the case of a resale opportunity). Agent 1 receives a (expected) payoff of $U_1(v_1, v_2) = v_1 - v_2$, while the (expected) payoffs of the other agents are zero; i.e., $U_j(v_1, v_2) = 0$, $j = 2, \ldots, n$. The expected total payment to the seller is $p(v_1, v_2) = p_1(v_1, v_2) + p_2(v_1, v_2) = v_2/2 + (v_2/v_1)(v_2/2) = v_2 (1 + v_2/v_1)/2 < v_2$, where p_i is the expected payment of agent i =1,2.

The previous results show that the outcome of an all-pay auction is not ex-ante efficient, since for the expected social welfare W the following inequalities hold $v_2 < W(v_1, v_2) = v_1 - v_2 + p(v_1, v_2) < v_1$.[6] From the perspective of the economic theory of lobbying, they illustrate the possibility that, even if the number of potential contenders is large, asymmetries among players might imply that the political rent is not fully dissipated (see Hillman and Riley, 1989: pp. 18-19). In addition, note that $\partial p/\partial v_1 < 0$ and $\partial p/\partial v_2 > 0$ ($p(\cdot)$ can be proved to be convex): indeed, Baye *et alii* (1993) show that a politician (the seller in the auction) wishing to maximize her revenue should be willing to select the two active lobbyists (the bidders) i^* and i^*+1 in order to maximize $p(v_i, v_{i+1})$. This implies that she might find it in her best interest to exclude lobbyists from 1 to i^*-1 from her "finalists short list", if she is allowed to (there is no point in excluding bidders from i^*+2 to n). This could be worthwhile for her because while the expected payment from any $i^* \neq 1$ in the finalist list is necessarily less than the payment expected from 1 in the largest auction, the expected payment from $i^* + 1$ may rise with respect to that of 2 and more than compensate the decrease of the other component of total payment.

This is the Exclusion Principle, which is intuitively based on the idea to raise (overall) incentives to spend for the active asymmetric participants by putting them on a more equal footing. More formally, the Exclusion Principle works by raising the equilibrium probability of winning of the least favourite contender (between the two who are active in equilibrium). From the perspective of economic theory of lobbying, Baye *et alii* (1993: p. 290) argues that the politician (the seller), under plausible circumstances, has an adverse incentive to preclude the lobbyists who most value the prize from participating in the lobbying game. The idea of handicapping the favourite is simple, interesting and it has some counterpart both in the auction literature with *incomplete* information[7] (if agents' valuations are not identically and independently distributed: see e.g. Myerson, 1981 and

[6] For the sake of simplicity, we assume that the seller's evaluation of the prize is zero.
[7] This means that the valuation of each bidder is private information to himself at the bidding stage: see e.g. Mas-Colell *et alii* (1995: section 23) for this terminology.

Klemperer, 2004: pp. 21-8) and in sport (e.g., in golf competitions).[8] However, note that bidder 1's exclusion decreases the expected social welfare.

Now consider an exogenously given positive reserve price. It turns out that the bidding Nash equilibrium is unique. In particular, if $v_1 > p_r \geq v_2$, the prize is efficiently allocated to bidder 1 for his bid of $r = p_r$, while the other bidders bid zero with probability 1 (if $v_1 = p_r$ bidder 1 is indifferent to receiving the prize and there is a continuum of Nash equilibria in which he bids $b_1 = 0$ with a positive probability). Things are more interesting if $p_r < v_2$. The relevant results are summarized in Proposition 1.

Proposition 1. *Consider an (first-price) all-pay auction with complete information (no resale possibility). Suppose* $v_2 \geq p_r \geq 0$. *Then, in the unique bidding Nash equilibrium: i)* $F_1(b_1) = b_1/v_2$ *on the support* $[p_r,v_2]$*; ii)* $F_2(b_2) = 1 - v_2/v_1 + b_2/v_1$ *on the support* $\{0 \cup [p_r,v_2]\}$*; iii)* $F_j(0) = 1, j = 3, \ldots, n$.

Proof: see Appendix 1. Note that Proposition 1 says that agent 2 randomises between $b_2 = 0$ and the uniform distribution on $[p_r,v_2]$ with probabilities of respectively $1 – (v_2 - p_r)/v_1$ and $(v_2 - p_r)/v_1$, and that agent 1 randomises between $b_2 = p_r$ and the uniform distribution on $[p_r,v_2]$ with probabilities of respectively p_r/v_2 and $1 - p_r/v_2$. The equilibrium cumulative distribution function of agents 1 and 2 are illustrated in Figure 1.

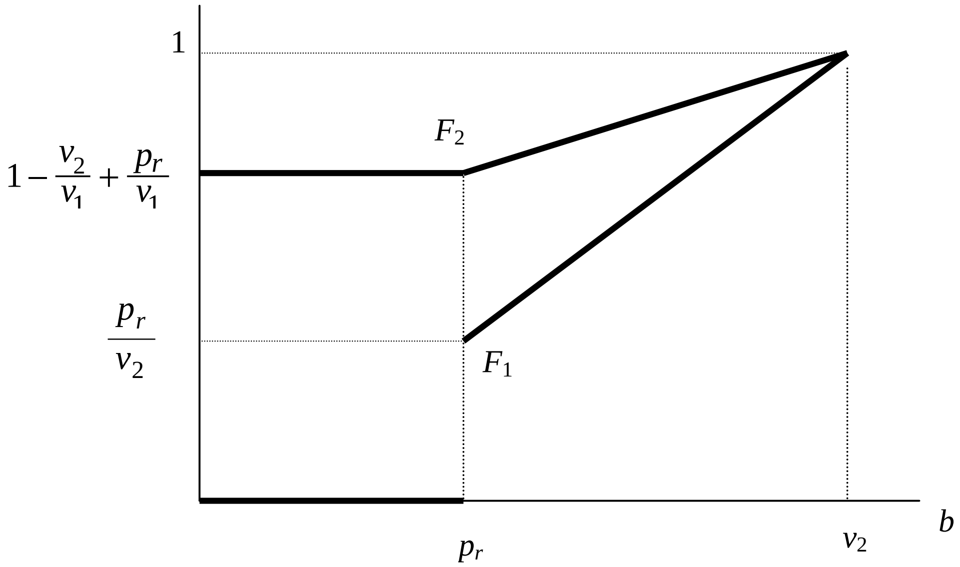

FIGURE 1: The equilibrium distribution functions for $p_r < v_2$

[8] Sport event organizers are typically interested in some "competitive balance" among players: a famous example comes from the history of the Giro d'Italia ("Tour of Italy"), the Italian most important cycling stage-race. It is reported that at the beginning of the twentieth century cyclist Alfredo Binda was so much stronger than his possible competitors (he had already won the Giro d’Italia five times) that the organisers paid him not to participate.

Also note that, exactly as in the case without a positive reserve price, agent 1 receives an (expected) payoff of $U_1(v_1, v_2, p_r) = v_1 - v_2$, while the (expected) payoffs of the other agents are zero. However, the prize is now won by agent 1 with probability $1 – (v_2^2 - p_r^2)/(2v_1v_2) > 1 - v_2/(2v_1)$; i.e., the introduction of a positive reserve price raises the probability of an ex-post efficient outcome by raising the probability that the prize is allocated to agent 1. Moreover, the expected total payment to the seller is given by ($v_2 \geq p_r$):

$$\tilde{p}(v_1, v_2, p_r) = \tilde{p}_1(v_1, v_2, p_r) + \tilde{p}_2(v_1, v_2, p_r) = \frac{v_2^2 + p_r^2}{2v_2} + \frac{v_2^2 - p_r^2}{2v_1} \qquad (1)$$

(note that $\tilde{p}(v_1, v_2, p_r)$ is continuous (and differentiable) for any $v_2 \geq p_r \geq 0$).[9] Equation (1) shows that, as it should be expected, the payment by agent 1 increases (on expectation), while the payment of agent 2 decreases, with respect to the case of a null reserve price. In particular, the increase is given by $(p_r^2/2)(v_2^{-1} – v_1^{-1})$: note that $\partial\tilde{p}/\partial p_r > 0$ and that $\tilde{p}(v_1, v_2, p_r)$ goes continuously from $p(v_1, v_2)$ to v_1 as p_r goes from 0 to v_1 ($\tilde{p} = 0$ if $v_1 < p_r$ and $\tilde{p} = p_r$ if $v_1 > p_r \geq v_2$). This is described in Figure 2. Note, finally, that since $W(v_1, v_2, p_r) = v_1 - v_2 + \tilde{p}(v_1, v_2, p_r)$, also the expected social welfare increases with respect to the case of a null reserve price.

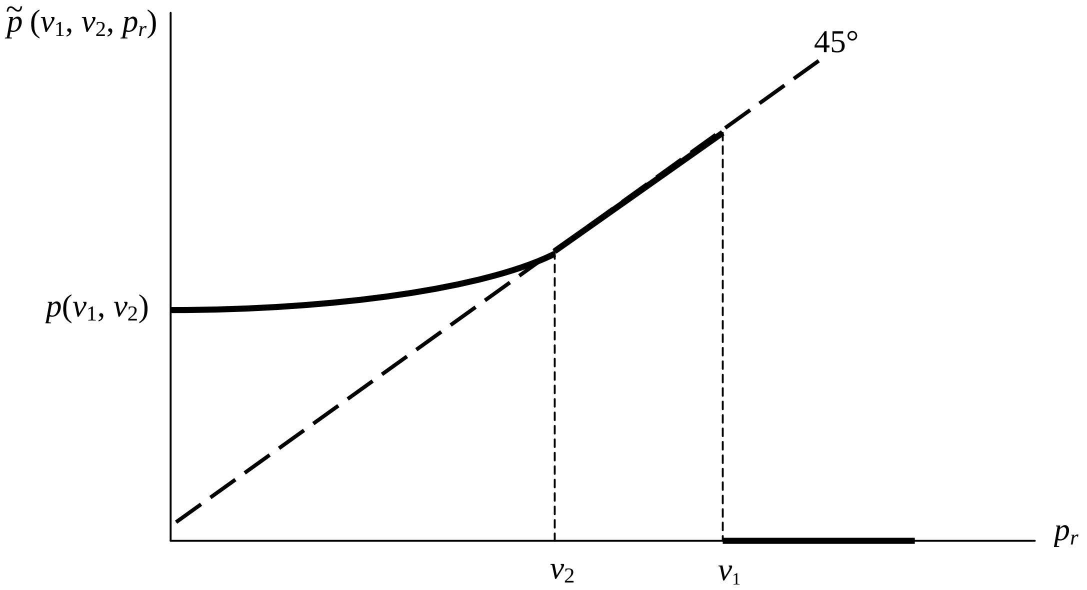

FIGURE 2: The expected revenue as a function of p_r

[9] For $v_1 = v_2 \geq p_r > 0$ there is more than a single Nash equilibrium, and equation (1) does not apply to all of them (see footnote 5).

By interpreting the reserve price *level* as a measure of the seller's bargaining power (following the suggestion by Milgrom, 1987), we can conclude that a fully-informed seller "strong" enough would use $p_r > v_2$, and in fact $p_r = v_1$ (i.e., she would make a take-it-or-leave-it offer to agent 1), without excluding any bidders. In such a case, there will not generally be full "rent dissipation" (unless in the extreme cases of either $v_1 = v_2$ or $p_r = v_1$), independently from the number of competitors. But, clearly, the Exclusion Principle does not apply, since it will always be better for the seller to use a large enough reserve price ($p_r \geq v_2$) rather than to exclude through a "finalists short list" some of the bidders who value most the prize.

However, the situation is more complex if the seller is *not* "strong enough" to set a reserve price $p_r \geq v_2$. Since $\partial\tilde{p}/\partial v_1 < 0$ and $\partial\tilde{p}/\partial v_2 > 0$, it is still possible that the exclusion of some agents is in her interest. In particular, she should choose i, j ($> i$) *and* p_r in order to maximize $\tilde{p}(v_i, v_j, p_r)$ under the "bargaining constraints" she faces. Notice that, if the reserve price that the seller can adopt does not depend on the agents she selects, she will always choose the largest possible reserve price, say p_r^+, and also agent i+1 when he chooses agent i: i.e., things are very much as in Baye *et alii* (1993), and a generalized version of the Exclusion Principle holds. But, in such a case, it cannot be strictly better to exclude agent from 1 to i - 1 if $p_r^+ \geq v_{i+1}$ (since $\tilde{p}(v_i, v_{i+1}, p_r) > p_r$ if $v_{i+1} > p_r$), which implies that there will be no profitable exclusion at all if $p_r^+ \geq v_3$. In other words, the case for the Exclusion Principle becomes weaker.

3. Exclusion for a seller facing incomplete information

The assumption that a fully informed seller can credibly exclude some bidder from her "short list" while she is unable to ask him a price not higher than his valuation does not seem particularly palatable as a general bargaining feature. For this reason in this section we refer to the setting introduced by Menicucci (2006) and Bertoletti (2008), and investigate the case of a seller facing incomplete information. In such a setting, $\tilde{p}(v_1, v_2, p_r)$ is the revenue the seller expects "ad interim" (before bidding takes place but after the definition of a possible "short list" of auction participants), where from her point of view v_1 and v_2 are respectively the first (highest) and the second (second-highest) order statistics of n stochastic variables (see e.g. Krishna, 2002: Appendix C). We generalize the assumptions of Bertoletti (2008) by assuming that v_1 and v_2 are jointly distributed on the support $[\underline{v}, \overline{v}]^2$, $\overline{v} > \underline{v} \geq 0$, according to a continuous density function $g(v_1, v_2)$ which is strictly positive for $\overline{v} > v_1 > v_2 > \underline{v}$.[10]

[10] Note that, obviously, $g(\cdot)$ depends on the joint distribution of the bidders' valuations: see Appendix 3 for the case of independent bidders' valuations.

It follows that the seller should set the optimal reserve price by maximizing with respect to p_r:

$$P^E(p_r) = \int_{\underline{v}}^{\bar{v}} \int_{v_2}^{\bar{v}} \tilde{p}(v_1, v_2, p_r) g(v_1, v_2) dv_1 dv_2$$

$$= \frac{p_r g_1(p_r)}{\gamma(p_r)} + \int_{p_r}^{\bar{v}} \int_{v_2}^{\bar{v}} \tilde{p}(v_1, v_2, p_r) g(v_1, v_2) dv_1 dv_2, \quad (2)$$

where $P^E(p_r) = E\{\tilde{p}(v_1, v_2, p_r)\}$ (the ad-interim expected value of $\tilde{p}$) is a continuous and differentiable function, $g_1(\cdot)$ is the density function of v_1, $\gamma(\cdot) = g_1(\cdot)/(G_2(\cdot) - G_1(\cdot))$ and $G_1(\cdot)$ and $G_2(\cdot)$ are the (marginal) distributions functions of respectively v_1 and v_2. Of course, $G_2(p_r) - G_1(p_r) = \text{Prob}\{v_2 < p_r < v_1\}$. We call γ the *generalized* hazard rate, since it is equal to the hazard rate $\lambda(\cdot) = h(\cdot)/(1 - H(\cdot))$, where $h(\cdot)$ is the density function which corresponds to distribution $H(\cdot)$, if the bidders' valuations are iid according to H. The role played by it in auction theory comes from the fact that $E\{v_1 - v_2\} = E\{1/\gamma(v_1)\}$, as it is easily seen (thus the expected value of the inverse of the generalized hazard rate measures the bidders' components of the ex ante expected social welfare). Note that $G_2 \geq G_1$ and thus $\gamma \geq 0$, and lim $\gamma(v) \to \infty$ for $v \to \bar{v}$. Also note that $P^E(\bar{v}) = 0$, and $P^E(\underline{v}) > 0$.

Computation shows that:

$$\frac{dP^E(p_r)}{dp_r} = -g_1(p_r)\varphi(p_r) + p_r \int_{p_r}^{\bar{v}} \int_{v_2}^{\bar{v}} \left(\frac{1}{v_2} - \frac{1}{v_1}\right) g(v_1, v_2) dv_1 dv_2 \,, \quad (3)$$

where $\varphi(\cdot) = (\cdot) - 1/\gamma(\cdot)$. We call φ the *generalized* virtual value, because it satisfies the property $E\{\varphi(v_1)\} = E\{v_2\}$ which is possessed by the "virtual value" studied by the literature on auction with incomplete information and iid valuations (see e.g. Krishna, 2002: section 5.2). Note that lim $\varphi(v) \to \bar{v}$ for $v \to \bar{v}$. Since $g_1(\underline{v}) = 0$, (3) implies that $dP^E(\underline{v})/dp_r > 0$ if $\underline{v} > 0$; moreover, $dP^E(\underline{v})/dp_r = 0$ if $\underline{v} = 0$ but in such a case $\varphi(\underline{v}) < 0$. Thus the optimal reserve price p_r^* is larger than $\underline{v}$, smaller than $\bar{v}$, and must satisfy $dP^E(p_r^*)/dp_r = 0$.

Notice that p_r^* depends on the set of the short-listed participants to the auction through $g(\cdot)$. Thus, in general, the optimal reserve price should be expected to change after a change in the short list. In particular, intuition suggests that the seller could prefer to exploit the opportunity to set an higher reserve price rather than exclude the contender who is (expected) to be "more eager" to buy. Unfortunately, it appears impossible to solve explicitly $dP^E(p_r)/dp_r = 0$ even in the simplest case in which the seller regards bidders' valuations (here denoted by v^j, where $j = 1, \dots, n$ indicates the bidders whose valuations are not ex ante ordered) as iid according to a uniform distribution on the support [0,1], and thus $g(v_1, v_2) = (n^2 - n)(v_2)^{n-2}$ (see Appendix 3 and e.g. Krishna, 2002: p. 267).

In spite of this difficulty, some results can be gained even discharging the reserve price optimal adjustment. Indeed, a solvable case arises if, as in Bertoletti (2008), the seller regards bidders' valuations as iid according to a continuous cumulative distribution function $H(\cdot)$ on the support $[\underline{v}, \bar{v}]$. The following Proposition 2 holds.

Proposition 2. *Consider an all-pay auction with complete information among bidders and any given reserve price. Suppose that the bidders' valuations are ex-ante iid according to a strictly increasing distribution H(·) with a continuous density function and a monotonic increasing hazard rate. In this case the seller maximizes her expected revenue by getting the largest possible number of actual participants*.

Proof: see Appendix 3 (it extends Bertoletti, 2008 to the case of a positive reserve price). Proposition 2 shows that no Exclusion Principle can apply if the seller regards the bidders' valuations as iid according to a monotonic hazard rate (a technical condition, equivalent to the "log-concavity" of $[1 - H(\cdot)]$, satisfied by many distributions: see Bagnoli M. and Bergstrom, 2005). This implies that $E\{v_1 - v_2\}$ decreases with respect to the number of (ex-ante identical) bidders and this, in turn, implies that a larger n cannot harm the seller (for a discussion see Bertoletti, 2008), *whatever* is the reserve price adopted. Thus, in this sense, the result of Menicucci (2007) holds under seemingly severe conditions.

However, we believe that a fair assessment of the Exclusion Principle in this setting should rather refer to the case of valuations which are not identically distributed. It is intuitive and easy to see that, if potential bidders' valuations are independent, associated to a bidders' group there are distributions for the first and the second order statistics such that they first-order stochastically dominate the corresponding distributions associated to any (strictly smaller) bidder sub-group (see Appendix 3 and Shaked and Shanthikumar, 1994: section 1.B.4). We speculate that, as in the case of iid bidders' valuations, a key question is the effect of enlarging the set of bidders on $E\{v_1 - v_2\}$,[11] but this appears hard to characterize in the general case, since the enlargement also affects the generalized hazard rate γ. In particular, $1/\gamma$ is an average of the different hazard rate $1/\lambda^j = (1 - H^j)/h^j$ whose (variable) weights are the normalized values of the so-called *reverse* hazard rates $\sigma^j = h^j/H^j$: see Appendix 3. Clearly, even monotonicity of the generalized hazard rate would not be enough to guarantee that $E\{v_1 - v_2\}$ decreases while the set of (independent but not identical) bidders enlarges.

A simpler case arises if there are only two types of bidders, say s and w, with $H^s(\cdot) = (H^w(\cdot))^\theta$,

[11] Note that, up to the second term of its Taylor expansion with respect to v_1, $\tilde{p} \approx v_2 - (v_1 - v_2)[1 - (p_r/v_2)^2]/2$.

$\theta > 1$: following Krishna (2002: section 4.3), we call them the strong (s) and the weak (w) bidder types, since H^s likelihood-ratio stochastically dominates H^w (see e.g. Krishna, 2002: Appendix B and Shaked and Shanthikumar, 1994: section 1.C). In such a case it is easy to see that the weights in the average that defines the generalized hazard rate are constant (with respect to its argument) and equal respectively to $\theta n^s/(\theta n^s + n^w)$ and $n^w/(\theta n^s + n^w)$ (where n^s and n^w are the numbers of the strong and the weak bidders, with $n = n^s + n^w$), and that the addition of any type of bidder generates a distribution for v_1 such that $g_1(v_1; n + 1)$ (with obvious notation) likelihood-ratio stochastically dominates $g_1(v_1; n)$: see (A.9) in Appendix 3. Moreover, the monotonicity of λ^w then implies the monotonicity of λ^s and is sufficient to guarantee the monotonicity of γ.

Assuming that λ^w is monotonic, a *sufficient* condition for getting a decrease $E\{v_1 - v_2\}$ by the addition of one *strong* bidder is:

$$H^s(v) < [\frac{d}{dv}\{1/\gamma(v;n^s,n^w)\}]/[\frac{d}{dv}\{1/\gamma(v;n^s+1,n^w)\}], \qquad (4)$$

as it can be seen by taking the difference of the expected values of the generalized hazard rate before and after the addition, and integrating by parts. Since computation shows that $d(1/\lambda^w)/dv > d(1/\lambda^s)/dv$ if λ^w is monotonic, it is clear that (4) is satisfied and accordingly that adding a "strong" independent bidder to the set of the auction participants does decrease the expected difference of $\{v_1 - v_2\}$. One can prove that such an addition actually decreases γ, and simulations for θ, n^s and n^w using the uniform distribution on [0,1] for H^w indeed suggest that it also always (whatever p_r) increases P^E, as in the iid valuations case. That is, the Exclusion principle appears not to extend to even such a case.

6. Conclusions.

It is known that, without a reserve price, (first-price) all-pay auctions are inefficient trade mechanisms under complete information among bidders. In this paper we have characterized their (mixed strategy) Nash equilibrium under a *positive* reserve price. As it is intuitive, a fully informed seller would like to set a reserve price (if she can credibly do that), since this increases overall efficiency and is profitable for her. Indeed, the ability to set a positive reserve price (to commit to refuse to sell) is obviously an important part of the seller's bargaining power. We have shown that, even with a positive reserve price, a fully-informed seller might find it better to exclude a bidder who is especially eager "to buy" (an extension of the so-called Exclusion Principle). However, we have also showed that, once the possibility that the reserve price is optimally set is taken into account, the case for the Exclusion Principle becomes weaker.

We have also argued that an appealing model should assume that the seller is not fully informed when setting her optimal reserve price and/or considering exclusion, and characterized her optimal reserve price for an all-pay auction with complete information of the bidders. In this setting, we have extended a result due to Bertoletti (2008) and showed that, if the seller regards the bidders' valuations as iid according to a monotonic hazard rate, the Exclusion principle cannot apply. Preliminary results for the case of independent but ex-ante asymmetric valuations (perhaps a more apt setting) seem to restrict the case for it to situations in which reserve prices cannot be used.

Appendix 1

Proof of Proposition 1. Clearly, for each agent i the set of (weakly) undominated strategies is given by $\{0 \cup [p_r,v_i)\}$. Moreover, it can be shown that in equilibrium no bidder plays $b_i \in (p_r,v_i)$, and no more than one agent bid p_r, with a positive probability. This is so because if at least two of them do the latter, both would have an incentive to move the probability mass slightly higher, so increasing their payoffs (the conditional probability of winning would jump, and so would the payoff). If exactly one agent i has a mass point at some $b_i \in (p_r,v_i)$, then no other agent would place density immediately below that bid (it would be better to move that density above the mass point). But then agent i would do strictly better by moving that mass down (see Che and Gale, 1998: p. 645, Lemma 1, and Hillman and Riley, 1989: pp. 22-23, Proposition 1, for a formal proof). Thus, all equilibrium cumulative distribution function $F_i(b_i)$ must be continuous on (p_r,v_i).

Now note that agent 1 can secure himself a payoff equal to $v_1 - v_2 > 0$ by bidding $b_1 = v_2$ with probability 1. It follows that his equilibrium strategy support cannot include $b_1 \in \{[0,p_r) \cup (v_2,v_1)\}$. Suppose that there is an agent $j \neq 1$ who gets in equilibrium a positive expected payoff. Then it must be the case that he bids $b_j > p_r$ with probability 1 (he cannot neither bid zero nor bid p_r with positive probability, because otherwise he would get respectively a null and a negative payoff, while he should be indifferent among all bids that belong to the support of his own equilibrium cumulative distribution function $F_j(b_j)$). And it must also be the case that his infimum bid does coincides with the infimum bid of agent 1, say b^-, because otherwise at least one of them would get a negative payoff by bidding his own infimum bid. In fact, we have found a contradiction, because even by bidding b^- at least one of them must get a negative payoff (the conditional probability of winning is zero). Thus no agent other than 1 can get a positive payoff in the equilibrium, or bid p_r with a positive probability.

In addition, any agent different from 1 bidding more than p_r with a positive probability must have an "infimum" bid ($\geq p_r$) not smaller than b^- (otherwise he would get less than zero from that bid), and at least one must bid b^- (otherwise it would pay to someone to move down some density). Similarly, at least two agents must share the maximum bid, say b^+, larger than p_r. Let us now suppose that two agents different from 1, say j and h, bid more than p_r with a positive probability. It must then be the case that:

$$v_j Prob(j\ wins | b_j = b) - b = v_j \Pi_{i \neq j}^n F_i(b) - b = v_h \Pi_{i \neq h}^n F_i(b) - b, \qquad \text{(A.1)}$$

for any $b > p_r$ belonging to the support of both $F_j(\cdot)$ and $F_h(\cdot)$. This implies that, for *all* such a b:

$$\frac{F_h(b)}{F_j(b)} = \frac{v_h}{v_j}, \qquad \text{(A.2)}$$

which implies that $F_h(\cdot)$ *strictly* first-order stochastically dominates $F_j(\cdot)$ if $j < h$. Let k the largest agent number among those bidding in equilibrium more than p_r with positive probability. This implies that the maximum bid larger than p_r belongs to the support of both $F_1(\cdot)$ and $F_k(\cdot)$. In turn, this implies that:

$$v_k Prob(k\ wins | b_k = b^+) - b^+ = v_k F_1(b^+) - b^+ = v_k - b^+ = 0, \qquad \text{(A.3)}$$

but then by bidding b^+ with probability 1 agent 2 would get a positive payoff, unless both $k = 2$ and $b^+ = v_2$.

It follows that in equilibrium only agent 1 and 2 are active, with agent 1 using $F_1(b_1)$ on a support $[b^-,v_2]$, while $F_2(b_2)$ possibly has support $\{0 \cup [b^-,v_2]\}$. Since it must be the case that for any $b \in [b^-,v_2]$:

$$v_2 F_1(b) - b = 0, \quad v_1 F_2(b) - b \geq v_1 - v_2, \qquad \text{(A.4)}$$

we can conclude that $b^- = p_r$, that $F_2(b_2) = 1 - v_2/v_1 + b_2/v_1$ has in fact the support $\{0 \cup [p_r,v_2]\}$, and that agent 1 uses $F_1(b_1) = b_1/v_2$ on the support $[p_r,v_2]$. *Q.E.D.*

Appendix 2

Proof of Proposition 2. Since the density function of the joint distribution of the first and second order statistics (see Appendix 3 and e.g. Krishna, 2002: p. 267) of n independent draws from H is given by:

$$g(v_1, v_2) = (n^2 - n)(H(v_2))^{n-2} h(v_1) h(v_2) I_{(v_2,\infty)}(v_1) \qquad \text{(A.5)}$$

(where $I_{(\cdot)}(\cdot)$ is the appropriate indicator function), the density function of v_1 conditional on v_2 is given by:

$$g_{1|2}(v_1|v_2) = \frac{g(v_1, v_2)}{n(n-1)(1-H(v_2))(H(v_2))^{n-2} h(v_2)} = \frac{h(v_1)}{1-H(v_2)} \qquad \text{(A.6)}$$

on the support $[v_2, \bar{v}]$ (note that it does not depend on n). Clearly, $E\{\tilde{p}(v_1, v_2, p_r)\} = E_{v_2}\{E_{v_1|v_2}\{\tilde{p}(v_1, v_2, p_r)\}\}$, with obvious notation for the previous expectations. Now consider, for any given p_r, the function $t(\cdot)$:

$$t(v_2, p_r) = E_{v_1|v_2}\{\tilde{p}(v_1, v_2, p_r)\} = \begin{cases} \int_{v_2}^{\bar{v}} \tilde{p}(v_1, v_2, p_r) \frac{h(v_1)}{1-H(v_2)} dv_1 & \text{for } v_2 \geq p_r, \\ p_r \frac{1-H(p_r)}{1-H(v_2)} & \text{for } v_2 \leq p_r, \end{cases} \qquad \text{(A.7))}$$

and note that it is continuous, and differentiable for any $p_r \neq v_2$. Clearly, $t(\cdot)$ increases with respect to v_2 if $p_r > v_2$.

Now consider the case $p_r \leq v_2$ and compute the derivative

$$\begin{aligned} \frac{\partial t(v_2, p_r)}{\partial v_2} &= \int_{v_2}^{\bar{v}} \Big[\frac{\partial \tilde{p}(v_1, v_2, p_r)}{\partial v_2} \frac{1}{1-H(v_2)} + \frac{h(v_2)\tilde{p}(v_1, v_2, p_r)}{[1-H(v_2)]^2}\Big] h(v_1) dv_1 \\ &\quad - \frac{h(v_2)\tilde{p}(v_2, v_2, p_r)}{1-H(v_2)}. \end{aligned} \qquad \text{(A.8)}$$

Then, by using the convexity of $\tilde{p}(\cdot)$ with respect to v_1:

$$\begin{aligned} \frac{\partial t(v_2, p_r)}{\partial v_2} &\geq \frac{1}{1-H(v_2)}\Big\{\int_{v_2}^{\bar{v}} \frac{\partial \tilde{p}(v_1, v_2, p_r)}{\partial v_2} h(v_1) dv_1 + \frac{h(v_2)}{1-H(v_2)} \int_{v_2}^{\bar{v}} \Big[v_2 + (v_1 - v_2)\Big(\frac{p_r^2}{2v_1^2} - \frac{1}{2}\Big)\Big] h(v_1) dv_1 - h(v_2) v_2\Big\} \\ &= \frac{1}{1-H(v_2)}\Big\{\int_{v_2}^{\bar{v}} \Big(\frac{1}{2} - \frac{p_r^2}{2v_1^2} + \frac{v_2}{v_1}\Big) h(v_1) dv_1 + \frac{h(v_2)}{2(1-H(v_2))}\Big(\frac{p_r^2}{v_1^2} - 1\Big)\int_{v_2}^{\bar{v}} (v_1 - v_2) h(v_1) dv_1\Big\} \qquad \text{(A.9)} \\ &= \int_{v_2}^{\bar{v}} \frac{v_2}{v_1} \frac{h(v_1)}{1-H(v_2)} dv_1 + \Big(\frac{1}{2} - \frac{p_r^2}{2v_1^2}\Big)\Big[1 - \int_{v_2}^{\bar{v}} \frac{\lambda(v_2)}{\lambda(v_1)} \frac{h(v_1)}{1-H(v_2)} dv_1\Big] = E_{v_1|v_2}\Big\{\frac{v_2}{v_1} + \Big(\frac{1}{2} - \frac{p_r^2}{2v_1^2}\Big)\Big(1 - \frac{\lambda(v_2)}{\lambda(v_1)}\Big)\Big\}. \end{aligned}$$

Thus $E_{v_1|v_2}\{p(v_1, v_2)\}$ is an everywhere increasing function of v_2 *if* the hazard rate is monotonic. Finally, since $G_2(v_2; n+1)$ first-order stochastically dominates $G_2(v_2; n)$ for any number n of bidders with independent valuations (where $G_i(v_i;\, n)$ is the distribution function of v_i, i = 1,2, for n draws from independent random variables: see Appendix 4), any reduction in n decreases the expected revenue of the seller *if* the hazard rate of $H(\cdot)$ is monotonic. *Q.E.D.*

Appendix 3

Consider the joint distribution of the first and second order statistics of n independent continuous random variables v^j, j = 1, …, n, whose distributions are indicated with $H^j(\cdot)$ ($h^j(\cdot)$ is the corresponding density function) on the common support $[\underline{v}, \overline{v}]$. Clearly, $G_1(v_1) = \prod_{j=1}^{n} H^j(v_1)$, and computation shows that:

$$
\begin{aligned}
G(v_1, v_2) &= G_1(v_2)[\sum_{i=1}^{n} \frac{H^i(v_1)}{H^i(v_2)} - (n-1)] \\
G_2(v_2) &= G_1(v_2)[\sum_{i=1}^{n} \frac{1}{H^i(v_2)} - (n-1)] \\
g_1(v_1) &= G_1(v_1) \sum_{i=1}^{n} \frac{h^i(v_1)}{H^i(v_1)} \qquad \text{(A.9)}
\end{aligned}
$$

$$
\begin{aligned}
g_2(v_2) &= G_1(v_2)[\sum_{i=1}^{n} \sum_{j=1}^{n}{}_{j \neq i} \frac{h^i(v_2)(1 - H^j(v_2))}{H^i(v_2) H^j(v_2)}] \\
g(v_1, v_2) &= G_1(v_2)[\sum_{i=1}^{n} \sum_{j=1}^{n}{}_{j \neq i} \frac{h^i(v_2) h^j(v_1)}{H^i(v_2) H^j(v_2)}] \\
G_2(x) - G_1(x) &= G_1(x)[\sum_{i=1}^{n} \frac{1}{H^i(x)} - n] \\
\frac{g_1(v_1; n+1)}{g_1(v_1; n)} &= H^{n+1}(v_1) + \frac{h^{n+1}(v_1)}{\sum_{i=1}^{n} \frac{h^i(v_1)}{H^i(v_1)}}.
\end{aligned}
$$

Note that $G_i(v_i;\, n)$ first-order stochastically dominates $G_i(v_i;\, m)$ for any $n > m$, i = 1,2, and that $1/\gamma$ is an average of the different $1/\lambda^j = (1 - H^j)/h^j$ whose weights are the values of the so-called *reverse* hazard rates weights $\sigma^j = h^j/H^j$ divided by $\Sigma\sigma^j$. Moreover, if the random variables are iid according to H and h, then $1/\gamma = (1 - H)/h$ and $g_1(v_i;\, n + 1)$ even likelihood-ratio stochastically dominates $g_1(v_i;\, n)$: see e.g. Krishna (2002: Appendix B) and Shaked and Shanthikumar (1994: section 1.C).

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