

This draft: October 2006

Some Notes on CE(S) Production Functions

PAOLO BERTOLETTI[§]

Dipartimento di economia politica e metodi quantitativi
Università di Pavia

Abstract

We reconsider the CES production functions, focusing on their additive separability properties and the constraints these impose on dual elasticities of *gross* substitution and complementarity. In particular, we offer some *alternative* characterizations of the (standard) homogeneous cases that seem to have gone unnoticed, as far as we know. We also start exploring the cases of the (additively separable) non-homothetic technologies that satisfy constancy conditions for the mentioned dual elasticities.

J.E.L. Classification: D11, D24, D33.

Keywords: Elasticity of substitution, CES, homotheticity.

[§] Facoltà di Economia, Via San Felice, 7, I-27100 Pavia, ITALY; Telephone: +39 0382 986202; Fax +39 0382 304226;
Email: paolo.bertoletti@unipv.it

1. INTRODUCTION

The Constant Elasticity of Substitution (CES) production function is an important member of the tool kit of microeconomics and appears in all textbooks. An early example was presented in Solow (1956), and the two-input form was extensively studied in Arrow et alii (1961). *Assuming* constant returns to scale, it can be written (see e.g. Blackorby and Russell, 1989):

$$f(\mathbf{x}) = \left[\sum_{i=1}^n \alpha_i x_i^\rho \right]^{1/\rho}, \quad (1)$$

where \mathbf{x} is a positive n -input vector, $\alpha_i > 0$, $i = 1, \dots, n$ and $0 \neq \rho \leq 1$, or

$$f(\mathbf{x}) = \beta \prod_{i=1}^n x_i^{\beta_i}, \quad (2)$$

in the special (Cobb-Douglas) case of $\rho = 0$, with $\beta, \beta_i > 0$ and $\sum_i \beta_i = 1$.

The “constant” elasticity of substitution (ES) is of course given by the parameter $\sigma = 1/(1 - \rho)$, which corresponds unambiguously to the original concept introduced by Hicks (1932: pp. 117 and 245). He noted that, for a given output y (i.e., along an isoquant), with only two inputs the effect of a change in the factor price ratio on the input ratio can be characterized by a scalar.¹ Several alternatives were proposed, as soon as Allen and Hicks (1934), for the generalization of this concept to the case of $n \geq 3$ inputs: see e.g. Uzawa (1962), and Blackorby and Russell (1981) and (1989). In particular, Blackorby and Russell (1989) convincingly argued that the so-called Morishima Elasticity of Substitution (MES), which reduces to the original Hicksian concept in the two-dimensional case, is the natural one, while the main alternatives suffer from some drawbacks. However, as shown by the same authors (Blackorby and Russell, 1981: theorems 3 and 4), for the CES production functions those different concepts coincides.

Over the fifty years since its introduction, a large literature has presented and analyzed functional forms that aim to *generalize* the CES technology, yielding less restrictive yet (hopefully) manageable (to estimate) demand functions: see e.g. Hanoch (1978) for an approach using the constancy of some ES measures. The purpose of these notes is much more limited in scope and just seeks out to offer some *alternative* characterizations of the standard CES production functions

¹ Hicks (1932) was actually concerned with the effect of changes in the input quantities on the distribution of income, but with only two inputs (and constant returns to scale) these two measures are just reciprocal of each other. Indeed, later Hicks (1970: pp. 289-90 and p. 296) gallantly attributed the elasticity of substitution to Joan Robinson (1933: pp. 256-9 and 330), and explicitly aimed to provide a dual concept, which is now known as the Hicks Elasticity of Complementarity (HEC): see e.g. Bertolotti (2005) and below.

which seem to have gone unnoticed, as far as we know. However, it is hoped that the perspective proposed here, in addition to deepening our understanding of the CES technology, may in a future suggest yet another gradual generalization of it of some interest. In particular, we do not make assumptions directly on the ES, but rather consider functional aspects that are related to it by the special properties of the CES production functions. Section 2 discusses the case of the Constant Elasticity of Complementarity (CEC) production functions, while section 3 takes up a dual approach based on the normalized profit function.

2. THE CES/CRRA/COST-CURVATURE PARAMETER

We start by stressing that the constant-returns-to-scale assumption in (1) and (2) is also due to the interest about growth and its distribution implications by Hicks, Solow and other early contributors to this literature. Indeed, while of course changing the returns to scale, and possibly making the resulting production functions non homogenous with respect to \mathbf{x} , any monotonic increasing transformation (MIT) of them would exhibit the same isoquants, and so, a fortiori, the same ES. Another reason is that in the parallel case of a CES *preference* ordering, obviously monotonic transformations of the production function do not matter at all. To illustrate this point, we follow Uzawa (1962) and consider the approach based on the consideration of the cost function $c(y, \mathbf{w})$ dual to any well-behaved production function $f(\mathbf{x})$, where $y = f(\mathbf{x})$ is the relevant level of output and the positive input price vector \mathbf{w} identifies the cost minimizing input bundle $\tilde{\mathbf{x}}(y, \mathbf{w})$ on the corresponding isoquant. Uzawa (1962) showed that, as is by now well known, the ES between inputs i and j is given by (in what follows subscripts indicate partial derivatives with respect to input prices w_i , $i, j = 1, 2, i \neq j$, or output, which we assume to be well defined):²

$$\sigma_{i,j}(y, \mathbf{w}) = \frac{c_{ij}(y, \mathbf{w})c(y, \mathbf{w})}{c_i(y, \mathbf{w})c_j(y, \mathbf{w})}. \quad (3)$$

It has been proven (Lau, 1976, Blackorby, Primont and Russell 1978: theorem 4.9 and Blackorby and Russell, 1981: theorem 2) that the ES σ_{ij} are *constant* (i.e., independent from (y, \mathbf{w}) and the specific inputs i and j) for a technology if and only if the cost function $c(y, \mathbf{w})$ can be written either $c(y, \mathbf{w}) = (\sum_i v^i(y) w_i^r)^{1/r}$, where $0 \neq r \leq 1$ and $v^i(\cdot)$, $i = 1, \dots, n$, are “arbitrary” (positive)

² The measure indicated in (3) is known as the Allen-Uzawa (partial) ES (AUES), but as recalled above it does coincide with the MES in the CES case (the only exception is the case of the so-called “implicit” Cobb-Douglas technology: see next footnote). In what follows we will write σ instead of σ_{ij} while stressing the case in which it does not depend on the specific couple of inputs (and similarly for the other elasticities to be introduced below).

functions, or $c(y, \mathbf{w}) = h(y) \prod_{i=1}^n w_i^{\delta_i}$, where $h(\cdot)$ is a positive and monotonic increasing function, $\delta_i > 0$, $\sum_i \delta_i = 1$, in the (“generalized Cobb-Douglas”) case of $r = 0$. Clearly, $\sigma = 1 - r$. Notice that the previous structure implies that the CES cost function is *additively* separable with respect to prices, i.e. completely separable in the discrete partition $\{\{1\}, \dots, \{n\}\}$ of the input set: see Blackorby, Primont and Russell (1978: chapter 4). The latter condition is equivalent to the possibility of writing the cost function as $C(y, \sum_i c^i(y, w_i))$ (see Blackorby, Primont and Russell 1978: corollary 4.8.8), where $C(\cdot)$ is an increasing function and each $c^i(\cdot)$ is positive. In fact, a CES structure requires more than additive separability of the cost function with respect to \mathbf{w} , which is enough only to guarantee that the ES is independent from \mathbf{w} (i.e., constant along the isoquant).³

Moreover, it turns out that a fundamental ingredient to get the CES ordinary production function representation (a MIT of (1) or (2)) is the condition of homotheticity.⁴ As it is well known, a technology represented by $y = f(\mathbf{x})$ is homothetic if and only if its cost function can be written $c(y, \mathbf{w}) = \hat{c}(\mathbf{w})h(y)$, where $\hat{c}(\mathbf{w})$ would be the unit cost for the linearly homogeneous production function $h(f(\cdot))$. Thus homotheticity crucially implies the additional separability of y from the input prices, and so $\sigma_{ij}(\mathbf{w}) = (\hat{c}_{ij}(\mathbf{w})\hat{c}(\mathbf{w})) / (\hat{c}_j(\mathbf{w})\hat{c}_i(\mathbf{w}))$ does not depend on y . In fact, under homotheticity the previous “implicit” CES cost function becomes $c(y, \mathbf{w}) = h(y)(\sum_i v_i w_i^r)^{1/r}$, which allows an explicit representation in terms of a production function (a MIT of (1)) also in the case of $\sigma \neq 1$.⁵ Indeed, under homotheticity (and some regular assumptions)⁶, it is known that a production function belongs to the CES class if and only if it is “additively separable”, i.e., if it can be written as a MIT of an *additive* production function $\sum_i f^i(x_i)$. Basically, overall homotheticity makes the additive separability of the cost function or of the production function a condition equivalent to the CES structure; similarly, the joint additive separability of the production function and of the cost function implies overall homotheticity, and then a CES structure: see Blackorby, Primont and Russell (1978: chapter 4, theorems 4.11-13), and Blackorby and Russell (1981: theorems 9-10).

³ Indeed, additive separability of the cost function implies (see Blackorby, Primont and Russell, 1978: theorem 4.9) that it must have a so-called “implicit CES or Cobb-Douglas structure”. That is, either $c(y, \mathbf{w}) = \Gamma(y)(\sum_i c^i(y, w_i)^{r(y)})^{1/r(y)}$ or $c(y, \mathbf{w}) = \Gamma(y) \prod_i c^i(y, w_i)^{\delta_i(y)}$, where each $c^i(\cdot) > 0$ is monotonic with respect to y and linearly homogeneous with respect to w_i , $0 \neq r(\cdot) \leq 1$, $\Gamma(\cdot)$, $\delta_i(\cdot) > 0$ and $\sum_i \delta_i(\cdot) = 1$. Notice that the AUES is constant in the *implicit* Cobb-Douglas case, but not so the MES, that is not symmetric with respect to inputs i and j (Blackorby and Russell, 1981: theorem 1).

⁴ A function is homothetic if it can be written as a monotonic transformation of a linearly homogeneous function. Notice that the cost function, being linearly homogeneous, is necessarily homothetic with respect to prices.

⁵ Notice that the *generalized* Cobb-Douglas technology quoted above is homothetic.

⁶ We assume monotonicity and differentiability of the relevant “aggregator” and “transformation” functions.

The roles of homotheticity and additive separability for the CES technology can be illustrated as follows: suppose that a production function is additively separable and thus can be written:

$$f(\mathbf{x}) = F\left(\sum_{i=1}^n f^i(x_i)\right), \quad (4)$$

where $F(\cdot)$ is a positive MIT. Homotheticity implies that $f^{i'}(x_i)/f^{j'}(x_j) = f^{i'}(\lambda x_i)/f^{j'}(\lambda x_j)$, for any x_i and x_j , $i, j = 1, \dots, n$ ($i \neq j$), and any $\lambda > 0$. Differentiating with respect to λ and putting $\lambda = 1$ shows that it must be that:

$$\frac{f^{i''}(x_i)x_i}{f^{i'}(x_i)} = \frac{f^{j''}(x_j)x_j}{f^{j'}(x_j)}, \quad (5)$$

which in turn implies that: (i) all $f^i(\cdot)$ must be an affine transformation (AT) of a common function $f(\cdot)$; and (ii) that $f(\cdot)$ must exhibit a constant elasticity of the marginal product ($(f''(x)x)/f'(x)$), that is what is called in the theory of choice under uncertainty a constant coefficient of relative risk aversion (CRRA). I.e., $f(\cdot)$ must belong to the so-called CRRA family, with either $f(x) = x^\rho$ or $f(x) = \ln x$.

(i) and (ii) together imply that any additively separable homothetic production function can be written either:

$$f(\mathbf{x}) = F\left(\gamma + \sum_{i=1}^n \gamma_i x_i^\rho\right), \quad (6)$$

where $\text{sign}\{\gamma_i\} = \text{sign}\{\rho\}$, and $0 \neq \rho \leq 1$ are required for the monotonicity and quasi-concavity of $f(\cdot)$, or:

$$f(\mathbf{x}) = F\left(\phi + \sum_{i=1}^n \phi_i \ln x_i\right). \quad (7)$$

with $\phi_i > 0$. It is straightforward to check that (6) and (7) are just functionally identical to (1) and (2) if indeed $F(\cdot)$ is the inverse of an opportune AT of the relevant $f(\cdot)$.

The previous result for linearly homogeneous CES functions was first formally proven by Eichhorn (1974), as far as we know, who showed that (1) and (2) are *characterized* by the functional property of so-called quasilinearity (a real function $\varphi(\mathbf{x})$ is quasilinear if it can be written $\varphi(\mathbf{x}) = l^{-1}(a_0 + \sum_i a_i l(x_i))$, where a_i are real constants and $l(\cdot)$ an invertible function: also see Eichhorn, 1978: chapter 2). An economic intuition for such a result goes as follows: notice first that the cost function for an *additive* production function must be such that $c_{ij}(\mathbf{w}, y) =$

$(c_{iy}(\mathbf{w},y)c_{jy}(\mathbf{w},y))/c_{yy}(\mathbf{w},y)$. That is, additivity (of the production function) makes the output (“size”) derivatives of the cost function to *completely* determine the element of the (“Slutsky”) substitution matrix (it connects input substitutability to certain movements along the output expansion path). If the additive production function is also homothetic (i.e., CES), then $\sigma_{ij}(\mathbf{w}) = (h'(y))^2/(h(y)h''(y))$, where $h(\cdot)$ is the MIT that would make it linearly homogeneous. Clearly, it must be the case that both sides of the previous expression are constant and equal to σ , which is both the ES and the coefficient of risk aversion of $h^{-1}(\cdot)$. Thus $h(\cdot)$ must be the inverse (of an AT) of the CRRA function which characterizes the homothetic additive production function.

Notice that $\sigma = (h'(y))^2/(h(y)h''(y)) = c_{y(y,\mathbf{w})}^2/(c_{yy}(y,\mathbf{w})c(y,\mathbf{w}))$ is also a measure of cost curvature with respect to output for such a homothetic *additive* production function. In particular, let the output elasticity of cost and marginal cost be given in general respectively by $m(y,\mathbf{w}) = \partial \ln c(y,\mathbf{w})/\partial \ln y$ and $\zeta(y,\mathbf{w}) = \partial \ln c_y(y,\mathbf{w})/\partial \ln y$: then $(c_y)^2/(c_{yy}c) = m/\zeta$. Also notice that for a generic (homothetic) CES this cost curvature interpretation would of course only apply to the additive production function nested into it. However, since these production functions share the same isoquants, they must also have the same ES, which does not depend upon the MIT of the latter function. More generally, notice that for any additively separable production function (for which $y = f(\mathbf{x})$ can be written as in (4)) it must be the case that $c(y,\mathbf{w}) = c^a(t(y),\mathbf{w})$, where $c^a(\cdot)$ is the cost function associated to the nested production function and $t(\cdot) = F^{-1}(\cdot)$. It follows by homotheticity that $c(y,\mathbf{w}) = \hat{c}(\mathbf{w})h^a(t(y))$, where $c^a(y,\mathbf{w}) = \hat{c}(\mathbf{w})h^a(y)$, for any homothetic CES function, with $\sigma = (h^a(\cdot))^2/(h^a(\cdot)h^{a''}(\cdot))$. This suggests that for them a special relationship between the ES and the cost function curvature generally exists: indeed some manipulations show that in such a case it must be that $\zeta/m = 1/\sigma + (t''y)/(mt')$. Notice that the (homothetic) CES cost-function curvature would be constant if and only if also the second term on the right-hand side were constant (this would happen if $t(\cdot)$ were an AT of a CRRA function and the elasticity of cost were constant too, and obviously if $t'' = 0$).

Summarizing, any *homothetic* CES production function can be written either as a (6) or (7), where $F(\cdot)$ is a MIT;⁷ equivalently, they are also given by any MIT of either (1) or (2): see e.g. Eichhorn (1978: chapter 2). Here we are interested in offering alternative interpretations of the constant ES/relative-risk-aversion/cost-curvature parameter σ that characterizes those production functions. Consider the so-called Hicks Elasticity of Complementarity (HEC: Hicks, 1970), $\rho^H_{ij}(\mathbf{x})$ ($i, j = 1, \dots, n$):

⁷ Notice that this result is somehow well known in the macroeconomics literature, which use the CRRA “instantaneous” utility function to generate constant elasticity of inter-temporal substitution under the assumption of additive separability over time (see e.g. Blanchard and Fischer, 1989: chapter 2).

$$\rho_{ij}^H(\mathbf{x}) = \frac{f_{ij}(\mathbf{x})f(\mathbf{x})}{f_i(\mathbf{x})f_j(\mathbf{x})}. \quad (8)$$

It can be shown that, if $f(\cdot)$ is concave, $\rho_{ij}^H(\mathbf{x}) = \eta_{ij}(\mathbf{x})/\gamma_j(\mathbf{x})$, where $\eta_{ij}(\mathbf{x}) = \partial \ln \underline{w}_i(\mathbf{x}) / \partial \ln x_j$ is the cross-elasticity of the *inverse* unconditional input demand $\underline{w}_i(\mathbf{x})$, and $\gamma_j(\mathbf{x}) = x_j \underline{w}_j(\mathbf{x}) / f(\mathbf{x})$ is the (normalized) output share of input j : see e.g. Bertolotti (2005). Note that the HEC is just a measure of the curvature of the overall “transformation” function if the production function is additively separable, and it is obviously null if it is additive. Similarly to what happens with the ES (see e.g. Blackorby, Primont and Russell, 1978: chapter 7 and Blackorby and Russell, 1981: section 3), the equality of certain HEC is equivalent to certain functional properties: see Kim (1997). In particular, consider the following property E:

$$(E) \quad \rho_{ik}^H(\mathbf{x}) = \rho_{jk}^H(\mathbf{x}), \quad i, j, k = 1, \dots, n, \quad i \neq k \neq j. \quad (9)$$

The following propositions hold.

PROPOSITION 1. *A production function is additively separable if and only if E holds.*

PROPOSITION 2. *Suppose that a technology is homothetic. Then the production function is CES if and only if E holds.*

PROPOSITION 3. *Suppose that a production function is homogeneous of degree $\sigma > 0$. Then it is a CES production function if and only if $\rho^H = 1 - (\sigma - 1)/(\sigma)$.*

Notice that Proposition 3 implies that the production functions given by (1) or (2) can be characterized as linearly homogeneous CEC production functions with $\rho^H = (1 - \rho)$. Indeed, the property of being CES or CEC is equivalent for any homogeneous production function. The intuition for Proposition 3 is straightforward and comes directly from our previous discussion. Consider the elasticity of scale $\xi(\mathbf{x}) = \partial \ln f(\lambda \mathbf{x}) / \partial \ln \lambda \big|_{\lambda=1}$ (a standard local measure of the returns to scale). It is well known that $m(y, \mathbf{w}) = 1/\xi(\tilde{\mathbf{x}}(y, \mathbf{w}))$ where $\tilde{\mathbf{x}}_i(y, \mathbf{w})$ is the cost-minimizing “conditional” demand of input i , and it can be shown that $\zeta(y, \mathbf{w}) = m(y, \mathbf{w}) - \partial \ln \xi(\tilde{\mathbf{x}}(y, \mathbf{w})) / \partial \ln y - 1$ ($\partial \ln \xi(\tilde{\mathbf{x}}(y, \mathbf{w})) / \partial \ln y$ is a measure of the change in marginal cost along the output expansion path). Under homogeneity of degree σ of the production function, $\zeta/m = (1/\sigma - 1)/(1/\sigma) = 1 - \sigma$. Now, for an additively separable production function $\rho^H = - (t''y)/t'$, where $t(\cdot) = F^{-1}(\cdot)$, and so it follows from our previous examination that for an homogeneous CES/CEC $1 - \sigma = 1/\sigma - \rho^H$, which is the result of Proposition 3.

More generally, a homothetic production function is CEC if and only if it can be written either as (6) or (7), where $F^{-1}(\cdot)$ is an AT of a CRRA function. Thus, up to a scalar multiplication, a (homothetic) CEC must be either a power function or an exponential function of an additive (homothetic) production function.⁸ Finally, notice that any homothetic CEC production function is also a homothetic CES production function (by Proposition 2), but not conversely. Consider for instance a logarithmic transformation of (1) (we would obviously get $\rho^H = 0$ for a logarithmic transformation of (2)): straightforward computations show that $\rho^H(\mathbf{x}) = (1 - \sigma)f(\mathbf{x})/\sigma$ for such a CES (see next section). Intuitively, the main difference between the ES and the HEC rests on fact that the latter also accounts for the technology returns to scale, which obviously have no relevance for the ES concept. Indeed, it follows from the definition of ρ_{ij}^H and the previous analysis that for any homothetic CES:

$$\rho^H(\mathbf{x}) = \frac{m(y)}{\sigma} - \zeta(y), \quad (10)$$

where $y = f(\mathbf{x})$. Notice that, due to homotheticity, in (10) m and ζ do not depend upon \mathbf{w} : thus ρ^H is also necessary constant along the relevant isoquant.

More generally, it can be shown (see Bertoletti, 2005: section 5) that, under homotheticity (and concavity of $f(\cdot)$), the following expression also holds:

$$\rho_{ij}^H(\mathbf{x}) = \xi(\mathbf{x})^{-1}(\rho_{ij}^A(y, \mathbf{x}) + \underline{\mu}(\mathbf{x})), \quad (11)$$

where $\underline{\mu}(\mathbf{x}) = \xi(\mathbf{x}) + \partial \ln \xi(\lambda \mathbf{x}) / \partial \ln \lambda \big|_{\lambda=1} - 1$ ($\partial \ln \xi(\lambda \mathbf{x}) / \partial \ln \lambda \big|_{\lambda=1}$ is a straightforward measure of the change in the returns to scale). $\rho_{ij}^A(y, \mathbf{x})$ (here evaluated at $y = f(\mathbf{x})$) is the so-called Antonelli Elasticity of Complementarity (AEC), a measure of complementarity which can be defined by making use of the distance function $d(y, \mathbf{x})$ and is dual to the ES: see Blackorby and Russell (1981: section 4). Notice that (11) implies that property E holds, under homotheticity (and concavity), if and only if an analogous property holds for the AEC. Moreover, ρ_{ij}^H depends upon i (or j) only if ρ_{ij}^A does. Finally, under homotheticity, the elasticity of scale is constant along an isoquant (as noted above), and the same can be proven to be true for $\underline{\mu}(\mathbf{x})$ (see Bertoletti, 2005: p. 191-2). So the HEC can vary (under homotheticity and concavity) for a fixed output level if and only if the AEC does. In particular, in the case of an homogeneous production function of degree o the returns to scale do not change, and so $\rho_{ij}^H(\mathbf{x}) = [\rho_{ij}^A(y, \mathbf{x}) - 1]/o + 1$ (if $o = 1$ then $\rho_{ij}^H(\mathbf{x}) = \rho_{ij}^A(y, \mathbf{x})$): thus the HEC are constant if and only if the AEC are constant too. Notice that for an homogeneous CES/CEC

⁸ By Proposition 1, any CEC function (possibly non-homothetic) must be additively separable.

production function $\sigma = 1/\rho^A$, which provides a characterization of this class of functions alternative to the one presented in Proposition 3 (for concave production functions)⁹. This is stated in the following proposition.

PROPOSITION 4. *Suppose that a production function is homogeneous and concave. Then it is a CES production function if and only if $\rho^A = 1/\sigma$.*

3. A DUAL APPROACH

An alternative approach to the CES production functions can be based on the consideration of the normalized profit function, $\pi(\underline{w})$ (see e.g. Lau, 1978):

$$\pi(\underline{w}) = \max_{x \geq 0} (f(x) - \underline{w}x), \quad (12)$$

where $\underline{w} = w/p$ is the vector of normalized input prices (p is the output price) and $f(\cdot)$ is assumed to be strictly concave. This profit function retains a very close relationship with the underlying production function, as shown e.g. by Lau (1978). In particular: a) the technology is homothetic if and only if both $f(\cdot)$ and $\pi(\cdot)$ are homothetic (Lau, 1978: theorem II-2); b) a homothetic production function is additively separable if and only if the normalized profit function is homothetic and additively separable (Lau, 1978: corollary 8.1); and c) the production function and the associated normalized profit function are both additively separable only if the technology is homothetic (Lau, 1978: theorem II-10). Thus a homothetic technology is CES if and only if its (normalized) profit function can be written as either

$$\pi(\underline{w}) = G\left(\sum_{i=1}^n \varepsilon_i \underline{w}_i^\mu\right)^{1/\mu}, \quad \text{or} \quad \pi(\underline{w}) = G\left(\varepsilon \prod_{i=1}^n \underline{w}_i^{\kappa_i}\right), \quad (13)$$

where $\varepsilon_i, \varepsilon, \kappa_i > 0, \sum_i \kappa_i = 1, 0 \neq \mu \leq 1$ and $G(\cdot)$ is a monotonic decreasing function. It is also easy to see that a production function is homogeneous of degree ρ if and only if the correspondent normalized profit function is homogeneous of degree $-\rho/(1 - \rho)$.

Consider the following Hotelling/Lau Elasticity of substitution (HLES: Lau, 1978: p. 197),¹⁰ $\sigma_{ij}^H(\underline{w})$:

⁹ Being based on the equality (10'), this argument is only valid under concavity (a condition assumed in Bertolotti, 2005), but we believe that the result also applies to any homogenous CES production function.

¹⁰ Also see Hanoch (1978: p. 293) for a generalization to the case with multiple outputs.

$$\sigma_{ij}^H(\underline{\mathbf{w}}) = -\frac{\pi_{ij}(\underline{\mathbf{w}})\pi(\underline{\mathbf{w}})}{\pi_i(\underline{\mathbf{w}})\pi_j(\underline{\mathbf{w}})}. \quad (14)$$

By Hotelling Lemma, $\sigma_{ij}^H(\underline{\mathbf{w}}) = \varepsilon_{ij}(\underline{\mathbf{w}})/\omega_j(\underline{\mathbf{w}})$, where $\varepsilon_{ij}(\underline{\mathbf{w}}) = \partial \ln x_i(\underline{\mathbf{w}}) / \partial \ln w_j$ is the cross-elasticity of the unconditional input demand of input i , $x_i(\underline{\mathbf{w}})$, and $\omega_j(\underline{\mathbf{w}}) = w_j x_j(\underline{\mathbf{w}}) / \pi(\underline{\mathbf{w}})$ is the profit share of input j . Clearly, it must be $\sigma^H = 0$ for an *additive* (strictly concave) production function (i.e., the inputs must be *gross* independent in such a case), and the HLES is more generally a measure of the profit function curvature if this is additively separable. It can be shown that, under homotheticity (and strictly concavity of $f(\cdot)$):¹¹

$$\sigma_{ij}^H(\underline{\mathbf{w}}) = (m(y) - 1) \left[\sigma_{ij}(\underline{\mathbf{w}}) - \frac{m(y)}{\zeta(y)} \right], \quad (15)$$

where $y = f(\mathbf{x}(\underline{\mathbf{w}}))$: see Bertoletti (2005). Note that, due to homotheticity, $\sigma_{ij}(\cdot)$ is independent from y ; moreover, as we know, $m/\zeta = (h'(y))^2 / (h(y)h''(y))$.

Let us define the following property E':

$$(E') \quad \sigma_{ik}^H(\underline{\mathbf{w}}) = \sigma_{jk}^H(\underline{\mathbf{w}}), \quad i, j, k = 1, \dots, n, \quad i \neq k \neq j. \quad (16)$$

(15) shows that (E') and (E) are equivalent properties for homothetic and (strictly) concave technology. Once again, the difference between the ES and the HLES rests on the account for the technology returns to scale. The following propositions hold.

PROPOSITION 5. *A normalized profit function is additively separable if and only if E' holds.*

PROPOSITION 6. *Suppose that a technology is homothetic and strictly concave. Then the production function is CES if and only if E' holds.*

Notice that, from (15), it must be the case that σ_{ij}^H does not depend on i (or j) if σ_{ij} does not, so in the CES case σ^H must be constant across inputs, but it might still vary according to the output level. It follows from the previous propositions and the same argument used to prove Proposition 3 that a technology is CES and homogeneous of degree $\sigma < 1$ if and only if the normalized profit function can be written either

$$\pi(\underline{\mathbf{w}}) = \left(\sum_i \varepsilon_i w_i^{(1+\sigma^H)\sigma/(1-\sigma)} \right)^{1/(1+\sigma^H)}, \quad \text{or} \quad \pi(\underline{\mathbf{w}}) = \varepsilon \prod_i w_i^{K_i}, \quad (17)$$

¹¹ Once again the ES appearing in (15) is the AUES: however, it coincides with the MES under conditions E or E' (see below) by the quoted results from Blackorby and Russell (1981).

where ε_i , $\varepsilon > 0 > \kappa_i$, and $\sum_i \kappa_i = -o/(1-o)$. Notice that in (17) σ^H_{ij} is constant, and indeed that also the following proposition holds.

PROPOSITION 7. *Suppose that a production function is homogeneous of degree o , $1 > o > 0$. Then it is a CES production function if and only if $\sigma^H = [(1-o)\sigma - 1]/o$, $i, j = 1, \dots, n$, $i \neq j$.*

Proposition 7 provides a further characterization of the homogeneous (strictly concave) CES/CEC technology, through constancy of the HLES: notice that $\rho^H/\sigma^H = -1/\sigma$. Also note that if $\sigma = 1$, then from Propositions 3, 4 and 7: $-\sigma^H = 1 = \rho^H = \rho^A$ for any homogeneous production function with $o < 1$ (i.e., in the Cobb-Douglas case). A constant HLES clearly requires, similarly to the case of a CEC technology, that up to a scalar multiplication, the normalized profit function must be either a power function or an exponential function of an additive profit function. Moreover, a constant σ^H implies a CES production function, but the conversely does not apply. To see the latter, consider again a logarithmic transformation of (1),¹² and notice that $m(y)/\zeta(y) = 1$ (since $m(y) = \zeta(y) = y$): thus $\sigma^H = (y-1)(\sigma-1)$ for such a non-homogenous CES function (remember that it is also $\rho^H = (1-\sigma)y/\sigma$).

In fact, the case in which $m(y)/\zeta(y)$ is constant (for a homothetic function) is interesting per se, since it holds if and only if $(h'(y))^2/(h(y)h''(y))$ is constant, where $\hat{c}(\mathbf{w})h(y)$ is the cost function for such a technology.¹³ Thus the underlying homothetic production function must be (an AT of) a CRRA function of a linear homogeneous function, as it is in the case in which the production function is itself homogeneous. Indeed, since under strict concavity $m(y, \mathbf{w})/\zeta(y, \mathbf{w}) = -\sum_i \varepsilon_{ji}(\mathbf{w})$ (see e.g. Bertolletti, 2005: p. 189), a constant m/ζ ratio implies that the demand system $x(\mathbf{w})$ is homogeneous of degree $\kappa = -m/\zeta$, so that the normalized profit function must be “close” to an homogeneous function of degree $\kappa + 1$. It can then be shown that, under E', also the normalized profit function must be (an AT of) a CRRA function of a linear homogeneous function of prices, and that the following proposition holds.

PROPOSITION 8. *A strictly concave (homothetic) production function is (up to an affine transformation) a CRRA function of (1) or (2) if and only if the normalized profit function is given by (13), where $G(\cdot)$ is (up to an affine transformation) a CRRA function too.*

¹² Obviously, also σ^H_{ij} would be zero for a logarithmic transformation of (2).

¹³ One can prove that in such a case $\underline{u}(\mathbf{x})$ is constant too, since $\underline{u}(\mathbf{x}) = -\zeta(f(\mathbf{x}))/m(f(\mathbf{x}))$ for a homothetic technology: see Bertolletti (2005: p. 192).

As an example, consider once again the case of a logarithmic transformation of (1): the cost function of such a CES technology is given by $c(y, \mathbf{w}) = \hat{c}(\mathbf{w})e^y$, where $\hat{c}(\mathbf{w}) = (\sum_i v_i w_i^r)^{1/r}$ (notice that in such a case $e^y = h^a(t(y)) = (e^{\rho y})^{1/\rho}$). Thus $\pi(\underline{\mathbf{w}}) = -\ln \hat{c}(\mathbf{w}) - 1$, where $-\ln \hat{c}(\mathbf{w}) = y(\underline{\mathbf{w}})$ is the output supply function.¹⁴

Finally, suppose that property E holds: then the (strictly) concave production function is additively separable, and by exploiting this property (14) can be generalized to encompasses the non-homothetic case (see e.g. Bertolotti, 2005: equation (11)) as follows:

$$\sigma_{ij}^H(\underline{\mathbf{w}}) = -\sigma_{ij}(y, \mathbf{w})(m(y, \mathbf{w}) - 1) \frac{\rho^H(\mathbf{x}(\underline{\mathbf{w}}))}{\zeta(y, \mathbf{w})}, \quad (18)$$

where $y = f(\mathbf{x}(\underline{\mathbf{w}}))$. Notice that (18) illustrates in a direct way that E' cannot hold as well, unless the underlying technology is homothetic and indeed CES, since both the normalized profit function and the cost function would also be additively separable.¹⁵ Thus the ES associated to a CEC production function cannot be either constant or just independent from the input prices unless the technology is homothetic. Similarly, if a technology exhibits constant HLES, it cannot satisfy also property E unless it is homothetic and in fact a homothetic CES. In the CES (homothetic) case (18) becomes:

$$\sigma^H(\underline{\mathbf{w}}) = (1 - m(y)) \frac{\rho^H(\mathbf{x}(\underline{\mathbf{w}}))}{\zeta(y)} \sigma, \quad (18')$$

which shows that for a homothetic CEC production function the HLES may still vary according to the output level (it must be constant along the isoquant). Conversely, even if the HLES is constant for a homothetic technology, the HEC may still vary according to y . However, it might well be that $(m - 1)$ and ζ are proportional, and so the ES, HEC and HLES are all constant for such a homothetic CE technology: a simple non-homogeneous example is given by $f(\mathbf{x}) = [\alpha_0 + \sum_i \alpha_i x_i^\rho]^{1/\rho}$, $\alpha_i > 0$, $i = 0, \dots, n$ and $0 \neq \rho \leq 1$, for which $(m - 1) = \alpha_0/(y^\rho - \alpha_0)$, $\zeta = (m - 1)/\sigma$ and $\rho^H = 1/\sigma = 1/\sigma^H$.

4. CONCLUSIONS

¹⁴ The corollary 4.1 in Lau (1978) appears invalid since it assumes that the primitive of a homogeneous function must be a homogenous function plus a constant (see Lau, 1978: theorem V-2).

¹⁵ The cost function is additively separable with respect to the input prices if and only if $\sigma_{ik} = \sigma_{jk}$, $i, j, k = 1, \dots, n$, $i \neq k \neq j$: see Blackorby, Primont and Russell (1978: theorem 7.4).

In this paper, under the assumption of homotheticity (whose importance in the present analysis cannot be exaggerated), we have reconsidered the case for CES production functions. The condition of being homothetic CES implies the additive separability of the production function, but being independent from the returns to scale it is per se obviously unable to pin down a specific functional form. In particular, it turns out that only under homogeneity the property of being a CES implies a well-defined form for the production functions, which are the familiar CES/Cobb-Douglas expressions. The intuitive reason we offer for this (well-known) result is that the constant ES parameter can be as well (and perhaps surprisingly) considered a parameter of cost-curvature for the additive production function nested into any homothetic CES. We have shown that, under homogeneity, this constant parameter is uniquely related with the measures of “size” curvature, which characterize both its (additively separable) production function and normalized profit function. A specific functional form also arises if, a bit more generally, the cost curvature with respect to output is constant.

Indeed, we have shown that equivalent functional forms emerge for the homogeneous CES if one assumes homogeneity and constancy of some related elasticities, namely the HEC (the AEC) or the HLES. This is so essentially because those elasticities incorporate equivalent separability requirements. Moreover, assuming constancy of the mentioned “gross” (i.e., without output compensation) elasticities HEC and the HLES imposes restraints on the returns to scale and defines sub-classes of CES production functions (power or exponential functions of additive homothetic production functions) that we might term constant elasticity (CE). Future work will perhaps investigate whether, also without homotheticity, those conditions are of interest, and which type of (additively-separable, non-CES) generalized technologies they characterized. For example, the so-called constant-ratio ES (CRES) introduced by Mukerji (1963) is clearly a non-homothetic CEC that also exhibits, from (18), constant ratios of the HLES.

Appendix

Proof of Proposition 1. Property E is obviously satisfied by any additively separable function. It implies that $f_{ik}(\mathbf{x})/f_i(\mathbf{x}) = f_{jk}(\mathbf{x})/f_j(\mathbf{x})$, $\forall \mathbf{x}$, which in turn can be shown to guarantee the property of additive separability of $f(\cdot)$: see e.g. Goldman and Uzawa (1964: theorem 1).

Proof of Proposition 2. Condition E implies that $f(\cdot)$ is additively separable, which together with homotheticity implies a (homothetic) CES structure. Conversely, suppose that the production

function is homothetic and CES: then it must be a MIT of either (6) or (7), and so it is additively separable and E holds by Proposition 1.

Proof of Proposition 3. A homogeneous of degree o CES production function is given by either (1) or (2) to the power of o , with ES respectively $\sigma < 1$ and 1. In turn, these power functions can be written respectively $(\sum_i f^i(x_i))^{(o\sigma)/(\sigma - 1)}$ or $e^{(\sum_i f^i(x_i))}$, which implies $\rho^H = 1 - (\sigma - 1)/(o\sigma)$. Conversely, suppose that the production function $f(\cdot)$ is homogenous of degree o and has a constant ρ^H . This implies that it is either $f(\mathbf{x}) = (\sum_i f^i(x_i))^{1/(1-\rho^H)}$ or $f(\mathbf{x}) = e^{(\sum_i f^i(x_i))}$. Moreover, the former property also requires that $f(\cdot)^{1/o}$ is linearly homogeneous and then, being additive, is either (1) or (2). I.e., either $f(\mathbf{x}) = (\sum_i \alpha_i x_i^{(1-\rho^H)o})^{1/(1-\rho^H)}$, $\alpha_i > 0$, or $f(\mathbf{x}) = \beta \prod_i x_i^{\beta_i}$, $\beta, \beta_i > 0$, $\sum_i \beta_i = o$, which are CES production function with $\sigma = 1/(1 - \alpha(1 - \rho^H))$.

Proof of Proposition 4. By Proposition 3, a homogeneous CES production function exhibits a constant HEC given by $\rho_{ij}^H = 1 - (\sigma - 1)/(o\sigma)$. By (11), this implies a constant AEC given by $\rho^A = 1/\sigma$, provided that $f(\cdot)$ is concave (this excludes a degree of homogeneity larger than 1). Conversely, under concavity and homogeneity of $f(\cdot)$ a constant AEC implies, by (11), a constant HEC given by $\rho^H = [\rho^A - 1]/o + 1$. From Proposition (3), this is equivalent to say that the production function must a CES with $\sigma = 1/\rho^A$.

Proof of Proposition 5. It follows from Lau (1978: theorem V-4).

Proof of Proposition 6. Together with homotheticity, condition E' implies that $f(\cdot)$ is homothetic and additively separable, which implies a (homothetic) CES structure. Conversely, suppose that the concave production function is homothetic and CES: then the normalized profit function must be homothetic and additively separable, and E' holds by Proposition 5.

Proof of Proposition 7. A homogeneous production function of degree o , $1 > o > 0$, is strictly concave. Moreover, $m = 1/o$ and $\zeta = (1 - o)/o$. So by (15) for a CES it must be $\sigma^H = [(1 - o)\sigma - 1]/o$, and conversely if $\sigma_{ij}^H = \sigma^H$ ($i \neq j$) the production function must be CES with $\sigma = (1 + o\sigma^H)/(1 - o)$.

Proof of Proposition 8. Suppose that the strictly concave production function is (up to an AT) a CRRA function of either (1) or (2): then $m/\zeta = -\kappa$ where κ is either -1 (if the CRRA is the logarithm) or $\kappa = 1/(a - 1)$ (if the relevant CRRA is the “ a ($0 < a < 1$) power function”). Then,

$\sum_j x_{ij}(\underline{w})w_j = \kappa x_i(\underline{w})$, $i, j = 1, \dots, n$, and $\mathbf{x}(\underline{w})$ is homogeneous of degree κ . Since $\mathbf{x}(\lambda \underline{w}) = \lambda^\kappa \mathbf{x}(\underline{w})$ for any $\lambda > 0$ (any \underline{w}), by integrating each $x_i(\lambda \underline{w})$ with respect to w_j and differentiating the result with respect to w_j ($j \neq i$) it can be shown that $\pi(\lambda \underline{w}) = \lambda^{\kappa+1} \pi(\underline{w}) + \chi(\lambda)$, where the latter is an arbitrary function of λ . Now notice that, by using homotheticity, $\pi(\lambda \underline{w}) = G(\lambda g^*(\underline{w}))$, where $g^*(\underline{w})$ is linearly homogeneous. Putting $z = g^*(\underline{w})$ this gives the functional equation $G(\lambda z) = \lambda^{\kappa+1} G(z) + \chi(\lambda)$, which was shown by Eichhorn (1978: section 2.7) to have only two solutions (for a monotonic $G(\cdot)$): either (an AT of) the logarithm of z , if $\kappa + 1 = 0$, or (an AT of) $z^{\kappa+1}$ if $\kappa + 1 < 0$. Thus the (normalized) profit function must be given by (13), where $G(\cdot)$ is (an AT of) the logarithm if $\kappa = 1$, or (an AT of) the $\kappa + 1$ power function. The converse follows because with $\pi(\underline{w})$ given by (13), where $G(\cdot)$ is (an AT of) a CRRA function, the (homothetic and additively separable) normalized profit function is either homogeneous (up to an additive constant) or an AT transformation of the logarithm of an homogeneous function. Thus it generates by Hotelling Lemma a homogeneous demand system $\mathbf{x}(\underline{w})$ which implies a constant m/ζ ratio. Accordingly, the underlying CES production function must be given by (an AT of) a CRRA transformation of either (1) or (2).

REFERENCES

- Allen, G.D.R. and J.R. Hicks, 1934, A reconsideration of the theory of value, Part II, *Economica* 1, 196-219.
- Arrow, K.J., Chenery, H.B., Minas, B.S. and R.M. Solow, 1961, Capital-labor substitution and economic efficiency, *Review of Economics and Statistics* 63, 225-250.
- Bertoletti, P., 2005, Elasticities of substitution and complementarity: A synthesis, *Journal of Productivity Analysis* 24, 183-196.
- Blackorby, C. and R.R. Russell, 1981, The Morishima elasticity of substitution; symmetry, constancy, separability, and its relationship to the Hicks and Allen elasticities, *Review of Economic Studies* 48, 147-58.
- Blackorby, C. and R.R. Russell, 1989, Will the real elasticity of substitution please stand up? (A comparison of the Allen/Uzawa and Morishima elasticities), *American Economic Review* 79, 882-888.
- Blackorby, C., Primont, D. and R.R. Russell, 1978, *Duality, separability, and functional structure: Theory and applications* (North-Holland: New York).
- Goldman, S.M. and H. Uzawa, 1964, A note on separability in demand analysis, *Econometrica* 32, 387-399.

- Eichhorn, W., 1974, Characterization of the CES production functions by quasilinearity, in W. Eichhorn, R. Henn, O. Opitz and R.W. Shephard, *Production Theory* (Springer-Verlag: Berlin), 21-33.
- Eichhorn, W., 1978, *Functional equations in economics* (Addison-Wesley Publishing Company: Reading, MA).
- Hanoch, G., 1978, Polar functions with constant two factors-one price elasticities, in M. Fuss and D. McFadden, *Production economics: a dual approach to theory and applications*, vol. 1 (North-Holland: Amsterdam), 287-309.
- Hicks, J.R., 1932, *The theory of wages* (McMillan: London).
- Hicks, J.R., 1970, Elasticities of substitution again: substitutes and complements, *Oxford Economic Papers* 22, 289-296.
- Kim, H.Y., 1997, Functional separability and elasticities of complementarity, *American Journal of Agricultural Economics* 79, 1177-1181.
- Lau, L.J., 1976, A note on elasticity of substitution functions, *Review of Economics Studies* 43, 353-358.
- Lau, L.J., 1978, Applications of profit functions, in M. Fuss and D. McFadden, *Production economics: a dual approach to theory and applications*, vol. 1 (North-Holland: Amsterdam), 133-216.
- Mukerji, V., 1963, A generalized S.M.A.C. function with constant ratios of elasticity of substitution, *Review of Economic Studies* 30, 233-236.
- Robinson, J., 1933, *The economics of imperfect competition* (McMillan: London).
- Solow, R.M., 1956, A contribution to the theory of economic growth, *Quarterly Journal of Economics* 70, 65-94.
- Uzawa, H., 1962, Production functions with constant elasticities of substitutions, *Review of Economic Studies* 4, 291-299.