Business Cycle models: A BC model without Capital
Advanced Macroeconomics II
Business Cycle Model without Capital

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Business Cycle Models

- The business cycle model without capital
  - Derivation of the competitive equilibrium
  - Log-linearization
  - Model Evaluation: IRFs to different shocks
  - Fiscal Policy

- The business cycle model with capital
  - Derivation of the competitive equilibrium
  - Log-linearization
  - Model Evaluation: IRFs to different shocks
  - Fiscal Policy
A simple microfounded discrete time monetary model, with:

1. Two agents: Households and Firms
2. All markets are perfectly competitive
3. All agents are *price-taker*
4. In the simple version of the classical monetary model money is considered a unit of account \(\rightarrow\) *cashless economy*
5. No capital accumulation.
HOW TO SOLVE THE MODEL - SEVEN STEPS

1. Microfoundations: solve agents’ optimal problems \( \Rightarrow \) find all the first order conditions.
2. Calculate the economy steady state
3. Log-linearize the model around the steady state
4. Write the system of \( n \) of difference equations \( \Rightarrow \) Solve for the recursive law of motion
5. Calculate the IRFs in response to different shocks
6. Calculate the moments: correlations, and standard deviations for the different variables both for the artificial economy and for the actual economy.
7. Compare how well the model economy matches the main features of the actual economy
STEP 1. Household Problem

- The households’ lifetime utility depends on: Consumption $C_t$ and Leisure $L_t$;

\[ E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, L_t) \text{ with } 0 < \beta = \frac{1}{1+\rho} < 1, \quad t = 0, 1, 2... \quad (1) \]

s.t.

\[ P_t C_t + Q_t B_t = B_{t-1} + W_t N_t + T_t + D_t \quad (2) \]

\[ N_t + L_t = 1 \text{ solvency constraint.} \]

- $U_C > 0, U_N < 0, U_{CC} \leq 0, U_{NN} \leq 0$
- $P_t$ is the price of good, $W_t$: nominal wage;
- $B_t$: quantity of one-period, nominally riskless bonds purchased in $t$ and maturing in $t+1$. Each bond pays 1 unit of money at maturity. Its price is $Q_t = \frac{1}{1+i_t} = \frac{1}{R_t}$, $R_t$: gross nominal interest rate.
- $T_t$: lump-sum transfer; $D_t$: Nominal profits (why? see next slides)
STEP 1. Solving the Langrangean:

\[ L = E_0 \sum_{t=0}^{\infty} \beta^t \{ U(C_t, N_t) - \lambda_t [P_t C_t + R_t^{-1} B_t - B_{t-1} - W_t N_t - T_t - D_t] \} \]

Labor supply:

\[ \frac{W_t}{P_t} = - \frac{U_N(t)}{U_C(t)} \]

Euler equation:

\[ R_t^{-1} = \beta E_t \left[ \left( \frac{P_t}{P_{t+1}} \right) \frac{U_C(t+1)}{U_C(t)} \right] \]
STEP 1. The period utility is separable and given by:

$$U(C_t, N_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\phi}}{1+\phi}$$

then optimality conditions becomes

$$\frac{W_t}{P_t} = N_t^\phi C_t^\sigma \quad (3)$$

$$R_t^{-1} = \beta E_t \left[ \left( \frac{1}{\Pi_{t+1}} \right) \left( \frac{C_{t+1}^{-\sigma}}{C_t^\sigma} \right) \right] \quad (4)$$

where $\Pi_{t+1} = \frac{P_{t+1}}{P_t}$.
STEP 1. Firms

- We abstract from capital accumulation and assume the following DRS technology:
  \[ Y_t = A_t F(N_t) \]  
  (5)

- with \( F_n > 0, F_{nn} \leq 0 \). Consider,
  \[ Y_t = A_t N_t^{1-\alpha} \]  
  (6)

- where \( \ln(A_t/A) = \hat{a}_t \) is an exogenous process of technological progress (or total factor productivity TFP), which evolves according to:
  \[ \hat{a}_t = \rho_a \hat{a}_{t-1} + \epsilon_{a,t}, \quad \epsilon_{a,t} \sim WN\left(0, \sigma^2_a\right) \ i.i.d. \]  
  (7)

- Since \( a_t \) is unknown agents must form expectations about it.
- Show that with decreasing return to scale firms profits are different from zero!
STEP 1. Firms Problem

\[
\begin{align*}
\max_{\{N_t\}} D_t & = P_t Y_t - W_t N_t \\
\text{s.t. } Y_t & = A_t N_t^{1-\alpha}
\end{align*}
\]

Foc:

\[
W_t \frac{P_t}{P_t} = (1 - \alpha) N_t^{-\alpha} A_t
\]

or marginal costs = Price, i.e. \( \frac{W_t}{(1-\alpha) N_t^{-\alpha} A_t} = P_t \). Notice:

\[
CT_t = W_t N_t (Y_t) = W_t \left( \frac{Y_t}{A_t} \right)^{\frac{1}{1-\alpha}}
\]

then

\[
MC_t = \frac{dCT_t}{dY_t} = \frac{W_t Y_t^{\frac{\alpha}{1-\alpha}}}{(1 - \alpha) A_t^{\frac{1}{1-\alpha}}} = \frac{W_t Y_t^{\frac{1-\alpha}{-\alpha}}}{(1 - \alpha) A_t^{\frac{1}{1-\alpha}} A_t} = \frac{W_t}{(1 - \alpha) N_t^{-\alpha} A_t}
\]
MARKET CLEARINGS

- Bonds Market
  \[ B_t = 0 \] \hspace{1cm} (9)

- Goods Market
  \[ C_t = Y_t \] \hspace{1cm} (10)

- Labor Market
  \[ N_t^\phi C_t^\sigma = (1 - \alpha) N_t^{-\alpha} A_t \] \hspace{1cm} (11)
STEP 2 - The steady state I

Euler  \[ \Pi = 1 \implies R = \frac{1}{\beta} \]

RC  \[ C = Y = AN^{(1-\alpha)} \implies N = \left( \frac{Y}{A} \right)^{\frac{1}{1-\alpha}} \]

LS  \[ \frac{W}{P} = N^{\phi} Y^{\sigma} = \left( \frac{Y}{A} \right)^{\frac{\phi}{1-\alpha}} Y^{\sigma} = \frac{Y^{\phi+(1-\alpha)\sigma}}{A^{\frac{\phi}{1-\alpha}}} \]

LD  \[ \frac{W}{P} = (1 - \alpha) AN^{-\alpha} = (1 - \alpha) A \frac{Y^{-\frac{\alpha}{1-\alpha}}}{A^{-\frac{\alpha}{1-\alpha}}} \]
STEP 2 - The steady state I

From Labor market equilibrium

\[
\frac{Y^{\frac{\phi+(1-\alpha)\sigma}{1-\alpha}}}{A^{\frac{\phi}{1-\alpha}}} = (1 - \alpha) \frac{Y^{-\frac{\alpha}{1-\alpha}}}{A^{-\frac{\alpha}{1-\alpha}}}
\]

then

\[
Y = (1 - \alpha)^{\frac{1-\alpha}{\phi+(1-\alpha)\sigma+\alpha}} A^{\frac{\phi+(1-\alpha)}{\phi+(1-\alpha)\sigma+\alpha}} = (1 - \alpha)^{\frac{1-\alpha}{\phi+(1-\alpha)\sigma+\alpha}}
\]

NOTE with \(A = 1\). Then

\[
C = Y = (1 - \alpha)^{\frac{1-\alpha}{\phi+(1-\alpha)\sigma+\alpha}}
\]

\[
N = (1 - \alpha)^{\frac{1}{(\phi+(1-\alpha)\sigma+\alpha)}}
\]

\[
\frac{W}{P} = (1 - \alpha)^{\frac{\phi+(1-\alpha)\sigma}{\phi+(1-\alpha)\sigma+\alpha}}
\]

RESULT 1: in the long-run, real variables do not depend on monetary variables \(\implies\) MONEY IS NEUTRAL in the LONG-RUN!!
STEP 3 - Log-linearization around the steady state

- Replace the dynamic non linear equations with dynamic linear equations
- Equations are linear in percentage deviations from the steady state

Define $x_t = \log X_t$,

and

$\hat{x}_t = \log \left( \frac{X_t}{X} \right)$, i.e. the log-deviations of $X_t$ from its steady state $X$. Thus $100*\hat{x}_t$ is approximately the percentage deviation of $x_t$ from its steady state. Take the first order approximation of $X_t$ around $x_t = X$. Note that:

$X_t = X e^{\log \left( \frac{X_t}{X} \right)} = X e^{\hat{x}_t}$ then:

$$
X_t = X e^{\hat{x}_t} \approx X e^{\hat{x}_t} \bigg|_{\hat{x}_t=0} + X e^{\hat{x}_t} \bigg|_{\hat{x}_t=0} (\hat{x}_t - 0) \\

\approx X (1 + \hat{x}_t)
$$
Examples:

1) $X_t = Y_t + Z_t$, then the steady state $\implies X = Y + Z$

log-linearization implies:

$$X (1 + \hat{x}_t) = Y (1 + \hat{y}_t) + Z (1 + \hat{z}_t),$$

simplifying for the SS:

$$\hat{x}_t = \frac{Y}{X}\hat{y}_t + \frac{Z}{X}\hat{z}_t.$$ 

2) $X_t = Y_t Z_t$:

$$Y_t = X_t Z_t \approx X (1 + \hat{x}_t) Z (1 + \hat{z}_t)$$
$$= XZ (1 + \hat{x}_t + \hat{z}_t + \hat{x}\hat{z}_t) = Y (1 + \hat{y}_t)$$

note that $\hat{x}_t\hat{y}_t$ is a second order term, then

$$Y (1 + \hat{y}_t)_t \approx XZ (1 + \hat{x}_t + \hat{z}_t)$$
$$\hat{y}_t \approx \hat{x}_t + \hat{z}_t$$

or taking the logs of the variables and subtracting the log of the SS we get the same result.
Examples:
3) \( Y_t = f(X_t) \)

\[
f(X_t) : Y_t = f(X_t) \approx f(X) \left( 1 + \frac{f'(X)}{f(X)} X \hat{x}_t \right) = Y \left( 1 + \hat{y}_t \right)
\]

Similarly
4) If \( Z_t = f(X_t, Y_t) \) :

\[
f(X_t, Y_t) = f(X, Y) \left( 1 + \frac{f_x(X, Y)}{f(X, Y)} X \cdot \hat{x}_t + \frac{f_y(X, Y)}{f(X, Y)} Y \hat{y}_t \right)
\]

and

\[
\hat{Z}_t = \frac{f_x(X, Y)}{f(X, Y)} X \hat{x}_t + \frac{f_y(X, Y)}{f(X, Y)} Y \hat{y}_t
\]
The Log-linearized model
Households labor supply (3) in log-terms: $\hat{c}_t \hat{n}_t$

$$\hat{w}_t - \hat{p}_t = \sigma \hat{c}_t + \phi \hat{n}_t \quad (12)$$

Log-linearizing the Euler equation (4) around the steady state:

$$\hat{c}_t = E_t \hat{c}_{t+1} - \frac{1}{\sigma} (\hat{i}_t - E_t \pi_{t+1}) \quad (13)$$

with $\hat{i}_t = \ln \left( \frac{R_t}{R} \right)$. Firms labor demand (8) in log-terms:

$$\hat{w}_t - \hat{p}_t = \hat{a}_t - \alpha \hat{n}_t \quad (14)$$

Firms production function (5) in log-terms:

$$\hat{y}_t = \hat{a}_t + (1 - \alpha) \hat{n}_t \quad (15)$$
Equilibrium I
Market Clearings

1. The feasibility constraint

\[ \hat{c}_t = \hat{y}_t \]  \hspace{1cm} (16)

2. Labor Market equilibrium and (16)

\[ \sigma \hat{y}_t + \phi \hat{n}_t = \hat{a}_t - \alpha \hat{n}_t \]  \hspace{1cm} (17)

The equilibrium level of output
using (15) and solving for \( \hat{y}_t \)

\[ \hat{y}_t = \frac{1 + \phi}{\sigma (1 - \alpha) + \phi + \alpha} \hat{a}_t \]  \hspace{1cm} (18)

Notice, with CRS, i.e. with \( \alpha = 0 \), then

\[ \hat{y}_t = \frac{1 + \phi}{\sigma + \phi} \hat{a}_t \]  \hspace{1cm} (19)
Equilibrium II

The equilibrium level of employment

From the production function:

\[ \hat{n}_t = \frac{(1 - \sigma)}{(\sigma (1 - \alpha) + \phi + \alpha)} \hat{a}_t \]  (20)

The equilibrium level of the real interest rate

Define \( \hat{r}_t \equiv \hat{a}_t - E_t \pi_{t+1} \).

From the log-linear Euler Equation + (16):

\[ \hat{r}_t = \sigma E_t \{ \Delta y_{t+1} \} = \sigma \psi_{ya} E_t \{ \Delta \hat{a}_{t+1} \}, \]  (21)

with \( \psi_{ya} = \frac{1 + \phi}{\sigma(1 - \alpha) + \phi + \alpha} \), or

\[ \hat{r}_t = - \frac{\sigma(1 + \phi)(1 - \rho_a)}{\sigma(1 - \alpha) + \phi + \alpha} a_t \]

equivalently

\[ r_t = \rho - \frac{\sigma(1 + \phi)(1 - \rho_a)}{\sigma(1 - \alpha) + \phi + \alpha} a_t \]
Equilibrium III

The equilibrium level of real wages

Define \( \hat{\omega}_t \equiv \hat{\nu}_t - \hat{p}_t \). From the labor demand:

\[
\hat{\omega}_t = \frac{\sigma + \phi}{\sigma (1 - \alpha) + \phi + \alpha} \hat{a}_t.
\] (22)
RESULT II: in the short-run, real variables do not depend on monetary variables $\implies$ MONEY IS NEUTRAL also in the SHORT-RUN!!

What happens to nominal variables? (Next Lectures)

- Nominal variables are: $\hat{i}_t, \hat{p}_t, \hat{m}_t, \hat{w}_t$
POSITIVE TECHNOLOGY SHOCK ($\sigma < 1$)
POSITIVE TECHNOLOGY SHOCK ($\sigma > 1$)
POSITIVE TECHNOLOGY SHOCK ($\sigma = 1$)
The Social Planner Problem

\[ \max U(C_t, N_t) \]
\[ s.t. \quad C_t = A_t F(N_t) \]

FOCs

\[ -\frac{U_{n,t}}{U_{c,t}} = A_t F_{n,t} \]

with the utility function:

\[ \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\phi}}{1+\phi} \]

\[ C_t^\sigma N_t^{\phi} = (1 - \alpha) A_t N_t^{-\alpha} \]

- The Social Planner Solution coincides with competitive solution
- Fluctuations of all variable (output, consumption, employment, investment...) are the optimal responses to exogenous changes in the economic environment (technology shocks, government spending shocks)
- Stabilization policies are not necessary