Notes on the Lucas Critique and the Cagan Inflation model

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1 Adaptive versus Rational Expectations

1) Adaptive Expectations

\[ E_t x_{t+1} = \lambda x_t + (1 - \lambda) E_{t-1} x_t \text{ con } 0 < \lambda < 1 \]

or

\[ E_t x_{t+1} = E_{t-1} x_t + \lambda (x_t - E_{t-1} x_t) \]

- agents update their expectation using information on past prediction errors.

2) Rational Expectations

\[ E_t [x_{t+1}|I_t] = x_{t+1} + u_{t+1} \]

- where \( u_{t+1} \) is a white noise. \( u_{t+1} \sim WN (0, \sigma_u^2) \), i.e., \( E_t u_{t+1} = 0 \) e \( Var (u_{t+1}) = \sigma_u^2 \) e \( Cov (u_t, u_{t+k}) = 0 \) for each \( k \neq 0 \).

- As a consequence agents cannot do deterministic errors. Indeed:

\[ E_t [E_t x_{t+1} - x_{t+1}] = E_t u_{t+1} = 0 \]

2 The policy evaluation Proposition: the Lucas critique (1976)

According to Lucas it is inappropriate to estimate the reduced form of economic models, if one proposes to use such models for the purpose of evaluating alternative economic policy regimes. Why? The reason is that the estimated parameters of such models would be a combination
of structural parameters together with other parameters that describe the characteristics of the policy rule itself (policy parameters). Given that agents use rational expectations, any change in policy regime would change the characteristics of the reduced-form response to policy variables. In other words, the parameters of the estimated reduced-form econometric models would be not invariant to changes in regime. Indeed, since agents’ expectations are contained in the parameters of the reduced-form, it follows that any change in policy regime will involve a change in these parameters.

Consider the following semi-reduced form of the model.

\[ y = \alpha_1 y^e + \alpha_2 x + \alpha_3 z \]  

where \( y \) represents any endogenous variable (as for example the aggregate demand), \( y^e \) is the rational expectation of \( y \), \( x \) is an exogenous variable (as for example the world demand for the domestic good), while \( z \) is a policy variable.

In order to obtain the reduce-form of the model, we need to solve (1) for the rational expectation of \( y \), and substitute this solution into (1). First, the rational expectation of equation (1) is:

\[ y^e = \alpha_1 y^e + \alpha_2 x^e + \alpha_3 z^e \]  

solving for \( y^e \)

\[ y^e = \frac{\alpha_2}{1 - \alpha_1} x^e + \frac{\alpha_3}{1 - \alpha_1} z^e \]  

We need to take the expectation of \( x \) and \( z \). The rational expectation hypothesis implies that agents know the equations of the two exogenous variables. In particular, we assume that they are described by the following equations

\[ x = x_{-1} + \varepsilon_x \]  

\[ z = \mu_1 z_{-1} + \mu_2 y_{-1} + \varepsilon_z \]  

where \( \varepsilon_x \) e \( \varepsilon_z \) are white noises. The rational expectation of (4) and (5) are then:

\[ x^e = x_{-1} \]  

\[ z^e = \mu_1 z_{-1} + \mu_2 y_{-1} \]  

since \( \varepsilon_x^e = 0 \) and \( \varepsilon_z^e = 0 \). Substituting (6) and (7) into (3) we get,

\[ y^e = \frac{\alpha_2}{1 - \alpha_1} x_{-1} + \frac{\alpha_3}{1 - \alpha_1} \mu_1 z_{-1} + \frac{\alpha_3}{1 - \alpha_1} \mu_2 y_{-1} \]  

where \( y^e \) is the rational expectation of endogenous variable. \( x \) and \( z \) are exogenous variables. \( \varepsilon_x \) and \( \varepsilon_z \) are white noises. The rational expectation of (4) and (5) are then:

\[ x^e = x_{-1} \]  

\[ z^e = \mu_1 z_{-1} + \mu_2 y_{-1} \]  

since \( \varepsilon_x^e = 0 \) and \( \varepsilon_z^e = 0 \). Substituting (6) and (7) into (3) we get,
substituting now (8) into (1) we get the reduced form of the model:

\[ y = \frac{\alpha_1 \alpha_2}{1 - \alpha_1} x_{-1} + \frac{\alpha_1 \alpha_3}{1 - \alpha_1} \mu_1 z_{-1} + \frac{\alpha_1 \alpha_3}{1 - \alpha_1} \mu_2 y_{-1} + \alpha_2 x + \alpha_3 z \]  

(9)

Econometricians estimate the following reduced form:

\[ y = c + \beta_1 x_{-1} + \beta_2 z_{-1} + \beta_3 y_{-1} + \beta_4 x + \beta_5 z + u_t \]  

(10)

where \( c \) is the constant term, \( \beta_1 = \frac{\alpha_1 \alpha_2}{1 - \alpha_1}, \beta_2 = \frac{\alpha_1 \alpha_3}{1 - \alpha_1} \mu_1, \beta_3 = \frac{\alpha_1 \alpha_3}{1 - \alpha_1} \mu_2, \beta_4 = \alpha_2, \beta_5 = \alpha_3 \) and \( u_t \) is the standard residual. The estimated parameters are: \( \hat{c}, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3, \hat{\beta}_4, \hat{\beta}_5 \). Notice that both \( \beta_2 \) and \( \beta_3 \) are function of both structural parameters and of policy parameters, i.e. \( \mu_1 \) and \( \mu_2 \). Then, for any regime switch, since agents use rational expectations they will update their knowledge of \( \mu_1 \) and \( \mu_2 \) and then \( \beta_2 \) and \( \beta_3 \) will differ from the estimated ones (\( \hat{\beta}_2 \) and \( \hat{\beta}_3 \)), which incorporate only past information. As a consequence, any policy evaluation of the regime switch done using \( \hat{\beta}_2 \) and \( \hat{\beta}_3 \) will be wrong. Notice however that, in the absence of a regime switch the estimation remain correct.

3 The Cagan model of Hyperinflation: adaptive versus rational expectation

Consider the following equation representing the equilibrium in the market of money (each variable is in log terms):

\[ m_t - p_t = -\alpha \left( p_{t+1}^e - p_t \right) \quad \text{con} \ \alpha > 0 \]  

(11)

where \( m \) is the money supply, \( p \) is the consumption price index; \( p_{t+1}^e \) is the expectation in \( t \) of \( p_{t+1} \). This means that \( p_{t+1}^e - p_t = \pi_{t+1}^e \) is the expected inflation rate. For simplicity, and without loss of generality, the money demand depending on output is assumed to be zero. Equation (11) can be rewritten as follows,

\[ p_t = \frac{1}{1 + \alpha} m_t + \frac{\alpha}{1 + \alpha} p_{t+1}^e. \]  

(12)

We now want to study the relationship between prices and money supply. In order to this, we need to do some assumption on the way agents form their expectations. We will consider two types of expectations: i) adaptive expectations and ii) rational expectations. Cagan (1956) suggested the adaptive expectations hypotheses. Sargent and Wallas (1973) converted the Cagan’s model into a rational expectation model.
3.1 Adaptive Expectations

If expectations are adaptive, then

\[ p_{t,t+1}^e = \lambda p_t + (1 - \lambda) p_{t-1,t}^e. \]  

(13)

We can solve equation (13) backwards. That is:

\[ p_{t-1,t}^e = \lambda p_{t-1} + (1 - \lambda) p_{t-2,t-1}^e \]  

(14)

substituting (14) into (13), we get

\[ p_{t,t+1}^e = \lambda p_t + (1 - \lambda) \left[ \lambda p_{t-1} + (1 - \lambda) p_{t-2,t-1}^e \right] \]  

(15)

\[ = \lambda p_t + (1 - \lambda) \lambda p_{t-1} + (1 - \lambda)^2 p_{t-2,t-1} \]  

(16)

iterating backward, we find \( p_{t-2,t-1}^e, p_{t-3,t-2}^e, p_{t-4,t-3}^e \) and so on...Substituting into (13), we can rewrite:

\[ p_{t,t+1}^e = \lambda \sum_{i=0}^{\infty} (1 - \lambda)^i p_{t-i} \]  

(17)

Substituting this result into (12):

\[ p_t = \frac{1}{1 + \alpha} m_t + \frac{\alpha}{1 + \alpha} \left[ \lambda p_t + \lambda \sum_{i=1}^{\infty} (1 - \lambda)^i p_{t-i} \right] \]

rearranging

\[ p_t \left( 1 - \frac{\alpha \lambda}{1 + \alpha} \right) = \frac{1}{1 + \alpha} m_t + \frac{\alpha}{1 + \alpha} \sum_{i=1}^{\infty} (1 - \lambda)^i p_{t-i} \]

and solving for \( p_t \)

\[ p_t = \frac{1}{1 + \alpha (1 - \lambda)} m_t + \frac{\alpha}{1 + \alpha (1 - \lambda)} \sum_{i=1}^{\infty} (1 - \lambda)^i p_{t-i} \]  

(18)

which is the solution of the model under adaptive expectation. Notice that the price level depends on the past prices and on the current value of money \( m \). Either a permanent or a transitory change in the money supply produces the same effects on the current price. Further, notice
that any policy announcement on the future supply of money has no effect on current prices. Current prices strongly depend on their past history.

We can rewrite the solution (18) in terms of inflation \( \pi_t = p_t - p_{t-1} \), and money growth rate \( \mu_t = m_t - m_{t-1} \). In this case:

\[
\pi_t = \frac{1}{1 + \alpha (1 - \lambda)} \mu_t + \frac{\alpha}{1 + \alpha (1 - \lambda)} \sum_{i=1}^{\infty} (1 - \lambda)^i \pi_{t-i}
\]

(19)

This equation simply states that, if the economy has experienced periods of very hyperinflation in the past, then the monetary authority has very low effect on current inflation, since agents do not forget the past. In other words, if the monetary policy was too accommodative toward inflation in the past, it becomes very hard to convince agents that the monetary regime has changed. Any announcement on the future policy has no effect on current inflation.

### 3.2 Rational Expectations

We now assume that expectations are rational. The starting point is again:

\[
p_t = \frac{1}{1 + \alpha} m_t + \frac{\alpha}{1 + \alpha} E_t p_{t+1}
\]

(20)

where we use the notation \( E_t p_{t+1} \) for indicating rational expectation in \( t \) for price in \( t + 1 \). Model with rational expectations are solved forward. Iterating forward (20), we find

\[
p_{t+1} = \frac{1}{1 + \alpha} m_{t+1} + \frac{\alpha}{1 + \alpha} E_{t+1} p_{t+2}
\]

(21)

taking its rational expectation,

\[
E_t p_{t+1} = \frac{1}{1 + \alpha} E_t m_{t+1} + \frac{\alpha}{1 + \alpha} E_t E_{t+1} p_{t+2}
\]

(22)

\( E_t E_{t+1} = E_t \) by the law of iterated expectation. Substituting (22) into (20), we get

\[
p_t = \frac{1}{1 + \alpha} m_t + \frac{\alpha}{1 + \alpha} \left[ \frac{1}{1 + \alpha} E_t m_{t+1} + \frac{\alpha}{1 + \alpha} E_t p_{t+2} \right]
\]

\[
= \frac{1}{1 + \alpha} m_t + \frac{\alpha}{1 + \alpha} \frac{1}{1 + \alpha} E_t m_{t+1} + \left( \frac{\alpha}{1 + \alpha} \right)^2 E_t p_{t+2}
\]

Iterating forward \( T \) times, we get the general solution:

\[
p_t = \frac{1}{1 + \alpha} \sum_{i=0}^{\infty} \left( \frac{\alpha}{1 + \alpha} \right)^i m_{t+i} + \left( \frac{\alpha}{1 + \alpha} \right)^T E_t p_{t+T}
\]
For $T \to \infty$, then $(\frac{\alpha}{1+\alpha})^T \to 0$ and the solution for the current price is:

$$p_t = \frac{1}{1+\alpha} \sum_{i=0}^{\infty} \left( \frac{\alpha}{1+\alpha} \right)^i m_{t+i}$$  \hspace{1cm} (23)

Past values of $p$ do not appear (23). Current prices are function of current and future values of the money supply. Any announcement on the future supply of money will affect current price immediately.

In terms of inflation and money growth rate, this implies that

$$\pi_t = \frac{1}{1+\alpha} \sum_{i=0}^{\infty} \left( \frac{\alpha}{1+\alpha} \right)^i \mu_{t+i},$$  \hspace{1cm} (24)

According to this equation, if the economy has experienced periods of very hyperinflation, then just one period is sufficient to strongly reduce the inflation rate. Central Bank announcements on its future policy affects the current value of inflation. The policy starts producing its effects before being implemented.

4 References

