EXOGENOUS GROWTH MODELS

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Goethe University 2011-2012
Course Outline

FIRST PART - GROWTH THEORIES

- Exogenous Growth
  - The Solow Model
  - The Ramsey model and the Golden Rule
- Introduction to Endogenous Growth models
  - The AK model - Romer (1990)
  - Two sector model of Endogenous growth

SECOND PART - BUSINESS CYCLE

- Introduction to NK model
  - The BMW model as a static approximation of a forward-looking NK model
  - The BMW model in a closed economy: inflation targeting versus Taylor rules
  - The BMW model in an open economy: comparisons with the Mundel Fleming model
TODAY

- Brief Review Growth Stylized Facts
- Introduction to the Solow Model
  - Derivations
  - The model performance and stylized facts
There is an enormous variation in the per capita income across economies. The poorest countries have per capita income that are less than 5 percent of per capita incomes in the richest countries.
# TABLE 1.1: Statistics on Growth and Development

<table>
<thead>
<tr>
<th>Country</th>
<th>GDP per capita, 1990</th>
<th>GDP per worker, 1990</th>
<th>Labor force participation rate, 1990</th>
<th>Average annual growth rate, 1960–90</th>
<th>Years to double</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>“Rich” countries</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>U.S.A.</td>
<td>18,073</td>
<td>36,810</td>
<td>0.49</td>
<td>1.4</td>
<td>51</td>
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<tr>
<td>West Germany</td>
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<td>29,488</td>
<td>0.49</td>
<td>2.5</td>
<td>28</td>
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<tr>
<td>Japan</td>
<td>14,317</td>
<td>22,602</td>
<td>0.63</td>
<td>5.0</td>
<td>14</td>
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<tr>
<td>France</td>
<td>13,896</td>
<td>30,340</td>
<td>0.46</td>
<td>2.7</td>
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</tr>
<tr>
<td>U.K.</td>
<td>13,223</td>
<td>26,767</td>
<td>0.49</td>
<td>2.0</td>
<td>35</td>
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<tr>
<td><strong>“Poor” countries</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>China</td>
<td>1,324</td>
<td>2,189</td>
<td>0.60</td>
<td>2.4</td>
<td>29</td>
</tr>
<tr>
<td>India</td>
<td>1,262</td>
<td>3,230</td>
<td>0.39</td>
<td>2.0</td>
<td>35</td>
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<tr>
<td>Zimbabwe</td>
<td>1,181</td>
<td>2,435</td>
<td>0.49</td>
<td>0.2</td>
<td>281</td>
</tr>
<tr>
<td>Uganda</td>
<td>554</td>
<td>1,142</td>
<td>0.49</td>
<td>-0.2</td>
<td>-281</td>
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<tr>
<td><strong>“Growth miracles”</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hong Kong</td>
<td>14,854</td>
<td>22,835</td>
<td>0.65</td>
<td>5.7</td>
<td>12</td>
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<tr>
<td>Singapore</td>
<td>11,698</td>
<td>24,344</td>
<td>0.48</td>
<td>5.3</td>
<td>13</td>
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<tr>
<td>Taiwan</td>
<td>8,067</td>
<td>18,418</td>
<td>0.44</td>
<td>5.7</td>
<td>12</td>
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<tr>
<td>South Korea</td>
<td>6,665</td>
<td>16,003</td>
<td>0.42</td>
<td>6.0</td>
<td>12</td>
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<tr>
<td><strong>“Growth disasters”</strong></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Venezuela</td>
<td>6,070</td>
<td>17,469</td>
<td>0.35</td>
<td>-0.5</td>
<td>-136</td>
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<tr>
<td>Madagascar</td>
<td>675</td>
<td>1,561</td>
<td>0.43</td>
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<tr>
<td>Mali</td>
<td>530</td>
<td>1,105</td>
<td>0.48</td>
<td>-1.0</td>
<td>-70</td>
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<tr>
<td>Chad</td>
<td>400</td>
<td>1,151</td>
<td>0.35</td>
<td>-1.7</td>
<td>-42</td>
</tr>
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</table>
## The Rich and the Poor

### GDP PER CAPITA 2000 and 2010

<table>
<thead>
<tr>
<th>Country</th>
<th>Code</th>
<th>2000</th>
<th>2010</th>
</tr>
</thead>
<tbody>
<tr>
<td>World</td>
<td>WLD</td>
<td>5271.446</td>
<td>9174.864</td>
</tr>
<tr>
<td><strong>Rich</strong> Countries</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>United States</td>
<td>USA</td>
<td>35081.92</td>
<td>47153.01</td>
</tr>
<tr>
<td>Germany</td>
<td>DEU</td>
<td>22945.71</td>
<td>40115.56</td>
</tr>
<tr>
<td>Japan</td>
<td>JPN</td>
<td>36789.22</td>
<td>42830.87</td>
</tr>
<tr>
<td>France</td>
<td>FRA</td>
<td>21828.29</td>
<td>39448.37</td>
</tr>
<tr>
<td>United Kingdom</td>
<td>GBR</td>
<td>25083</td>
<td>36343.25</td>
</tr>
<tr>
<td><strong>Poor</strong> Countries</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>China</td>
<td>CHN</td>
<td>949.1781</td>
<td>4428.463</td>
</tr>
<tr>
<td>India</td>
<td>IND</td>
<td>436.6476</td>
<td>1410.33</td>
</tr>
<tr>
<td>Zimbabwe</td>
<td>ZWE</td>
<td>534.7912</td>
<td>594.7018</td>
</tr>
<tr>
<td>Uganda</td>
<td>UGA</td>
<td>255.7806</td>
<td>508.9387</td>
</tr>
<tr>
<td><strong>Growth Miracles</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hong Kong SAR, China</td>
<td>HKG</td>
<td>25374.5</td>
<td>31756.91</td>
</tr>
<tr>
<td>Singapore</td>
<td>SGP</td>
<td>23814.56</td>
<td>41119.76</td>
</tr>
<tr>
<td>Korea, Rep.</td>
<td>KOR</td>
<td>11346.66</td>
<td>20756.69</td>
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<tr>
<td><strong>Growth Disasters</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Venezuela, RB</td>
<td>VEN</td>
<td>4818.708</td>
<td>13589.77</td>
</tr>
<tr>
<td>Madagascar</td>
<td>MDG</td>
<td>252.3761</td>
<td>420.9976</td>
</tr>
<tr>
<td>Mali</td>
<td>MLI</td>
<td>214.4666</td>
<td>601.9121</td>
</tr>
<tr>
<td>Chad</td>
<td>TCD</td>
<td>168.45</td>
<td>675.8416</td>
</tr>
</tbody>
</table>

Source: World Bank
World Population by GDP per worker

Figure 1.2: World Population by GDP per Worker, 1960 and 1988

Percentage of World Population

GDP per Worker, Relative to the U.S.
The world distribution of income in 1970
The world distribution of income in 2000
Rates of Economic Growth vary substantially across countries.
Growth rates are not generally constant over time. For the world as whole, growth rates were close to zero over most of the history but have increased sharply in the twentieth century. For individual countries, growth rates also change over time.
A country relative position in the world distribution of per capita incomes is not immutable. Countries can move from being poor to being rich, or vice versa.

EXAMPLE: Venezuela vs Italy.
## Fifteen Growth Miracles

<table>
<thead>
<tr>
<th>Country</th>
<th>Growth 1960-2000</th>
<th>Factor increase</th>
</tr>
</thead>
<tbody>
<tr>
<td>Taiwan</td>
<td>6.25</td>
<td>11.3</td>
</tr>
<tr>
<td>Botswana</td>
<td>6.07</td>
<td>10.6</td>
</tr>
<tr>
<td>Hong Kong</td>
<td>5.67</td>
<td>9.09</td>
</tr>
<tr>
<td>Korea, Republic of</td>
<td>5.41</td>
<td>8.24</td>
</tr>
<tr>
<td>Singapore</td>
<td>5.09</td>
<td>7.29</td>
</tr>
<tr>
<td>Thailand</td>
<td>4.50</td>
<td>5.83</td>
</tr>
<tr>
<td>Cyprus</td>
<td>4.30</td>
<td>5.39</td>
</tr>
<tr>
<td>Japan</td>
<td>4.13</td>
<td>5.04</td>
</tr>
<tr>
<td>Ireland</td>
<td>4.10</td>
<td>5.00</td>
</tr>
<tr>
<td>China</td>
<td>3.99</td>
<td>4.77</td>
</tr>
<tr>
<td>Romania</td>
<td>3.91</td>
<td>4.63</td>
</tr>
<tr>
<td>Mauritius</td>
<td>3.88</td>
<td>4.58</td>
</tr>
<tr>
<td>Malaysia</td>
<td>3.82</td>
<td>4.48</td>
</tr>
<tr>
<td>Portugal</td>
<td>3.48</td>
<td>3.93</td>
</tr>
<tr>
<td>Indonesia</td>
<td>3.34</td>
<td>3.72</td>
</tr>
</tbody>
</table>
### Fifteen Growth Disasters

<table>
<thead>
<tr>
<th>Country</th>
<th>Growth 1960-2000</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Peru</td>
<td>0.00</td>
<td>1.00</td>
</tr>
<tr>
<td>Mauritania</td>
<td>-0.11</td>
<td>0.96</td>
</tr>
<tr>
<td>Senegal</td>
<td>-0.26</td>
<td>0.90</td>
</tr>
<tr>
<td>Chad</td>
<td>-0.43</td>
<td>0.84</td>
</tr>
<tr>
<td>Mozambique</td>
<td>-0.50</td>
<td>0.82</td>
</tr>
<tr>
<td>Madagascar</td>
<td>-0.60</td>
<td>0.79</td>
</tr>
<tr>
<td>Zambia</td>
<td>-0.61</td>
<td>0.78</td>
</tr>
<tr>
<td>Mali</td>
<td>-0.77</td>
<td>0.74</td>
</tr>
<tr>
<td>Venezuela</td>
<td>-0.88</td>
<td>0.70</td>
</tr>
<tr>
<td>Niger</td>
<td>-1.03</td>
<td>0.66</td>
</tr>
<tr>
<td>Nigeria</td>
<td>-1.21</td>
<td>0.62</td>
</tr>
<tr>
<td>Nicaragua</td>
<td>-1.30</td>
<td>0.59</td>
</tr>
<tr>
<td>Central African Republic</td>
<td>-1.56</td>
<td>0.53</td>
</tr>
<tr>
<td>Angola</td>
<td>-2.04</td>
<td>0.44</td>
</tr>
<tr>
<td>Congo, Democratic Rep.</td>
<td>-4.00</td>
<td>0.20</td>
</tr>
</tbody>
</table>
In the US over the last century,

1. the real rate of return to capital, $r$, shows no trend upward or downward;

2. the shares of income devoted to capital, $rK/Y$, and the share of income devoted to labor, $wL/Y$, show no trend;

3. the average growth rate of output per person has been positive and relatively constant over time, i.e. the US exhibits steady, sustained per capita income growth.
Real Per Capita GDP in the US

Source: Maddison (1995) and author’s calculations.
Growth in output and growth in the volume of international trade are closely related.
Growth in trade and GDP 1960-1990
Both skilled and unskilled workers tend to migrate from poor to rich countries or regions

**ROBERT LUCAS:** this movements of labor tell us something about real wages. The returns of both skilled and unskilled
Kaldor (1963) listed a number of stylized facts that he thought typified the process of economic growth:

1. Per capita output grows over time, and its growth rate does not tend to diminish.
2. Physical capital per worker grows over time.
3. The rate of return to capital is nearly constant.
4. The ratio of physical capital to output is nearly constant.
5. The shares of labor and physical capital in national income are nearly constant.
6. The growth rate of output per worker differs substantially across countries.
The Solow Model

- Check for the ability of the model to explain the stylized facts
- Neoclassical model
  - Countries produce and consume one single good (units of GDP);
  - There is no international trade (since there is only one good)
  - Technology is exogenous
  - Perfect competition in all markets
The basic model is characterized by two equations:

1. a production function;
2. a capital accumulation equation.
The Solow Model

- The Neoclassical Aggregate Production Function

\[ Y(t) = F(A(t), K(t), L(t)) \]

where \( K(t) \) is physical capital, \( L(t) \) is labor and \( A(t) \) is a exogenous technology shift (TFP)

- Technology is free; it is publicly available as a non-excludable, non-rival good.
The Solow Model: Key Assumption

- **Assumptions**

\[ F_K(K, L, A) \equiv \frac{\partial F(\cdot)}{\partial K} > 0, \quad F_L(K, L, A) \equiv \frac{\partial F(\cdot)}{\partial L} > 0, \]

\[ F_{KK}(K, L, A) \equiv \frac{\partial^2 F(\cdot)}{\partial K^2} < 0, \quad F_{LL}(K, L, A) \equiv \frac{\partial^2 F(\cdot)}{\partial L^2} < 0. \]

- \( F \) exhibits **constant return to scale** in \( K \) and \( L \) \( \implies \) \( F \) is linear homogeneous (homogeneous of degree 1)
The Solow Model: Key Assumption

- **Capital accumulation**

  \[ \dot{K} = I - d \times K. \]

- \( I = sY \) where \( s \) represents a constant savings rate.
- \( d \) is capital depreciation rate
The Inada Conditions

\[ \lim_{K \to 0} F_K(\cdot) = \infty \text{ and } \lim_{K \to \infty} F_K(\cdot) = 0 \text{ for all } L > 0 \text{ and all } A \]

\[ \lim_{L \to 0} F_L(\cdot) = \infty \text{ and } \lim_{L \to \infty} F_L(\cdot) = 0 \text{ for all } K > 0 \text{ and all } A \]

Important in ensuring the existence of interior equilibria.
Firms’ profits maximization

\[
\max_{L(t) \geq 0, K(t) \geq 0} F[K(t), L(t), A(t)] - w(t) L(t) - R(t) K(t)
\]

FOCs

\[
w(t) = F_L[K(t), L(t), A(t)],
\]

\[
R(t) = F_K[K(t), L(t), A(t)].
\]

From the assumption of homogeneity of degree 1

\[
Y(t) = w(t) L(t) + R(t) K(t).
\]

and thus firms profits are zero!
The production function can be specified as follows:

\[ Y_t = K_t^\alpha (A_t L_t)^{1-\alpha} \quad \text{with} \quad 0 < \alpha < 1 \]

It is Cobb-Douglas production function with constant return to scale, where \( A \) is a technology variable labour augmenting. \( AL \) are the efficient units of labor. The rate of growth of technological progress is exogenous and defined as

\[
\frac{dA}{dT} \frac{1}{A} = \frac{d}{dt} \ln(A) = \frac{\dot{A}}{A} = g
\]
The Solow Model
The Solow Model

- The demand for capital implies
  \[ R_t = \alpha K_t^{\alpha-1} (A_t L_t)^{1-\alpha} \]
  and thus the capital share is
  \[ \frac{RK}{Y} = \alpha \frac{K^{\alpha-1} (AL)^{1-\alpha} K}{Y} = \alpha \text{ constant} \]

- The labor demand implies
  \[ w_t = (1 - \alpha) K_t^\alpha (A_t L_t)^{-\alpha} A_t \]
  and thus the labor share is
  \[ \frac{wL}{Y} = (1 - \alpha) \frac{K^\alpha (AL)^{-\alpha} AL}{Y} = 1 - \alpha \text{ constant} \]

- The labor share and the capital share are constant in the long run, in accordance with Kaldor stylized facts.
Production function in terms of output per worker:

\[ y = k^\alpha A^{1-\alpha} \]

where \( y = Y/L \) and \( k = K/L \)

Production function in terms of efficient unit of labor per worker is

\[ \tilde{y} = \tilde{k}^\alpha \]

where \( \tilde{y} = Y/(AL) \) and \( \tilde{k} = K/(AL) \).
The Solow Model

To see the growth implications of the model, we take the log and then differentiate $y = Ak^\alpha$, we find

$$\frac{\dot{y}}{y} = (1 - \alpha) \frac{\dot{A}}{A} + \alpha \frac{\dot{k}}{k}$$

notice however that $\frac{\dot{k}}{k} = \left(\frac{K}{L}\right) \frac{\dot{L}}{K}$. which is equal to

$$\left(\frac{K}{L}\right) \frac{L}{K} = \frac{L}{K} \frac{\dot{K}L - \dot{L}K}{L^2}$$

then

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L}$$

where the labor force growth rate is exogenous and given by $\frac{\dot{L}}{L} = n$. 
We can express capital in terms of efficient unit of labor per worker

\[ \tilde{k} = \frac{K}{AL} \]

then

\[ \frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} - \frac{A}{\bar{A}} \]
The Solow Model

- Remember that \( \dot{K} = I - d \times K \). This means that \( \frac{\dot{K}}{K} \) can be written as,
  \[
  \frac{\dot{K}}{K} = s\frac{Y}{K} - d
  \]
- If \( \frac{Y}{K} \) is constant, then \( \frac{\dot{K}}{K} \) is also constant.
- While from \( \frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} \) we can rewrite,
  \[
  \frac{\dot{K}}{K} = \frac{\dot{k}}{k} + \frac{\dot{L}}{L} = s\frac{Y/L}{K/L} - d
  \]
  then
  \[
  \dot{k} = sy - (d + n)k = sk^\alpha A^{1-\alpha} - (d + n)k
  \]
  is the law of motion of capital per worker, while
  \[
  \dot{\tilde{k}} = s\tilde{y} - (g + d + n)\tilde{k} = s\tilde{k}^\alpha - (g + d + n)\tilde{k}
  \]
  is the law of motion of capital per unit of efficient labor.
The Solow Model

An economy starts with a given stock of capital per worker $k_0$, a given population growth rate, $n$, a given technology growth rate, $g$, and a given investment rate.

1. How does output per worker, $y$, (or output per efficient units of labor per worker, $\hat{y}$) evolve over time?
2. How does the economy growth?
3. How does output per worker compare in the long-run between two economies that have different investment rate?
4. How does output per worker compare in the long-run between two economies that have different technology growth rate?
The Solow Model

- With $\dot{A}/A = 0$, then capital per worker is
  $$\dot{k} = sk^\alpha - (d + n) k$$
- In the steady state $\dot{k} = 0$ and
  $$k^* = \left(\frac{s}{d + n}\right)^{1/(1-\alpha)}$$
- Thus
  $$y^* = (k^*)^\alpha = \left(\frac{s}{d + n}\right)^{\frac{\alpha}{1-\alpha}}$$
- In the Solow model countries with higher savings/investment rate will tend to be richer, ceteris paribus. Such countries have more capital per worker and thus more output per worker.
- Countries with high population growth rate, in contrast will tend to be poorer according to the Solow model.
The Solow Model

- The Solow diagram determines the steady state value of output per worker.

- The dynamics converges to the steady state value of capital per workers $k^*$. 
The Solow Model

**Balanced growth.** Notice that \( \dot{k} = sk^\alpha - (d + n) k \)

\[
\begin{align*}
\frac{\dot{k}}{k} &= 0 \text{ implies } \frac{\dot{K}}{K} = \text{constant} \implies \frac{\dot{K}}{K} = n \\
\frac{\dot{y}}{y} &= 0 \text{ implies } \frac{\dot{Y}}{Y} = \text{constant} \implies \frac{\dot{Y}}{Y} = n \\
\frac{\dot{c}}{c} &= 0 \text{ implies } \frac{\dot{C}}{C} = \text{constant} \implies \frac{\dot{C}}{C} = n \\
\frac{\dot{K}}{Y} &= k^{1-\alpha} = \frac{s}{d + n} \text{ in steady state}
\end{align*}
\]
The Solow Model

The Solow diagram and household consumption
The Solow Model. The Golden rule

- Consumption is

\[ c_t = F(k_t) - sF(k_t) \]

- At the steady state consumption is

\[ c^* = F(k^*) - sF(k^*) = (k^*)^\alpha - (d + n) k^* \]

- Consumption is maximum if

\[ \frac{\partial c}{\partial k} = 0 : F'(k) - (d + n) = 0 \]
The Solow Model. The Golden rule

\( \frac{\partial c}{\partial k} = 0 \) implies

\[ \alpha (k)^{\alpha - 1} - (d + n) = 0 \]

and thus, solving for \( k \)

\[ k^{GR} = \left( \frac{\alpha}{d + n} \right)^{\frac{1}{1-\alpha}} \]

we call \( k^{GR} \) the Golden rule capital stock, that is the value of \( k \) such that \( \frac{\partial c}{\partial k} = 0 \) and consumption is maximum.
The Solow Model. The Golden rule
The Solow Model. The Golden rule

- **DYNAMIC EFFICIENCY.** If \( k^{GR} > k^*_1 \iff s_1 < s^{GR} \) and \( c^{*1} < c^{GR} \). Increasing savings increases also the steady state per capita consumption. However, by increasing savings, consumption decreases on impact.

- **DYNAMIC INEFFICIENCY.** If \( k^{GR} < k^*_2 \iff s_2 > s^{GR} \) and \( c^{*2} < c^{GR} \). The economy is oversaving and is said to be dynamically inefficient.
An increase in the investment rate $s$ results in the steady state value of capital per worker increasing.

- the steady state value of capital per worker increases.
The Solow Model

- An increase in population growth
The Solow Model

Consider now \( \frac{\dot{A}}{A} \neq 0 = g \). In the steady state \( \dot{k} = 0 \) and

\[
\tilde{k}^* = \left( \frac{s}{d + n + g} \right)^{\frac{1}{1-\alpha}}
\]

thus

\[
\tilde{y}^* = \tilde{k}^\alpha = \left( \frac{s}{d + n + g} \right)^{\frac{\alpha}{1-\alpha}}
\]

while output per worker is

\[
y^* (t) = \left( \frac{s}{d + n + g} \right)^{\frac{\alpha}{1-\alpha}} A(t)
\]

where \( t \) is included just to remind that \( A \) is an exogenous growing variable.
Consider now $\dot{A}/A \neq 0 = g$. The Solow diagram becomes
**The Solow Model**

- **Balanced growth.** Notice that \( \dot{k} = s\tilde{k}^\alpha - (g + d + n) \tilde{k} \)

\[
\begin{align*}
\frac{\dot{\tilde{k}}}{\tilde{k}} = 0 & \implies \frac{\dot{K}}{K} = g + n \quad \text{and} \quad \frac{\dot{k}}{k} = g \\
\frac{\dot{\tilde{y}}}{\tilde{y}} = 0 & \implies \frac{\dot{Y}}{Y} = g + n \quad \text{and} \quad \frac{\dot{y}}{y} = g \\
\frac{\dot{\tilde{c}}}{\tilde{c}} = 0 & \implies \frac{\dot{C}}{C} = g + n \quad \text{and} \quad \frac{\dot{c}}{c} = g
\end{align*}
\]

- Finally notice that given \( \frac{\dot{K}}{K} = \frac{\dot{Y}}{Y} \implies \frac{\dot{K}}{K} - \frac{\dot{Y}}{Y} = 0 \) and thus

\[
\frac{K}{Y} = \frac{s}{g + n + d} = \text{constant}
\]
THE BALANCED GROWTH PATH. In the Solow model the growth process follows a balanced growth path if the GDP per worker, consumption per worker, the real wage per worker and capital per worker, all grow at one and the same constant rate, \( g \), the labor force, i.e. population grows at a constant rate \( n \), GDP, consumption and capital grow at a common rate, \( g + n \), the capital-output ratio is constant and the rate of return on capital is constant.
The Solow Model

- An increase in the savings rate
The Solow Model

An increase in the savings rate and conditional convergence.
Output per worker along the balanced growth path is determined by technology, investment rate and the population growth rate.

Changes in the investment rate and the population growth rate affect the long-run level of output per worker, but do not affect its long-run growth rate.

Policy changes do not have long-run growth effects.

Policy changes can have level effects, that is a permanent policy change can permanently raise (or lower) the level of per capita output.

Conditional convergence.
The Ramsey Growth Model

The Solow model:
- The first general equilibrium model with production side.
- It is empirically testable.
- Lacks microfoundations (saving is not determined exogenously)
- Exogenous technological progress explains all.

The Ramsey Model
- Introduces endogenous savings/consumption decision
- Optimal consumption
- The decentralized equilibrium can be compared with the Pareto efficient equilibrium
The Ramsey Growth Model

The decentralized model

- Two agents
  - Households: maximize their life-time utility subject to an intertemporal budget constraint
  - Firms: maximize profits subject to their factor accumulation constraint

- General Equilibrium. Demand = Supply.

The Social Planner model

- The social planner maximizes the households’ utility subject to aggregate resource constraint of the economy.
- If the decentralized solution coincide with the Social Planner solution, then the outcome is Pareto optimal.
The Ramsey Growth Model

The Household Problem

- One infinitely-lived household maximizing the intertemporal utility

\[ \int_0^\infty u(c_t) L_t e^{-\rho t} dt = \int_0^\infty u(c_t) L_t e^{(n-\rho)t} dt \]

where \( \rho \) is the discount factor. A higher \( \rho \) implies a smaller desiderability of future consumption in terms of utility compared to utility obtained by current consumption.

- where \( u'(c) > 0 \) and \( u''(c) < 0 \) and \( \lim_{c \to 0} u'(c) = \infty \) and \( \lim_{c \to \infty} u'(c) = 0 \) (Inada Conditions)

- Population growth rate is constant (equal to \( n \)) and at time \( t = 0 \) there is only one individual in the economy (i.e. \( L_0 = 1 \)), so that the total population at any time \( t \) is given by \( L_t = e^{nt} \) and thus \( \frac{\dot{L}}{L} = n \).
The Ramsey Growth Model

The Household Problem

- The household budget constraint is

\[ \dot{B} = w_t L_t + r_t B_t - C_t \]

where \( B \) are assets, \( w \) is the wage rate, \( r \) is the interest rate. In per capita terms

\[ \dot{b} = w_t + (r_t - n) b_t - c_t \]
The Ramsey Growth Model

The Household Problem

- The trasversality condition

\[
\lim_{t \to \infty} \left( \begin{array}{c} b_t e^0 \\ \int_{0}^{t} (r_s - n) ds \end{array} \right) \geq 0
\]

- The present value of current and future assets must be asymptotically nonnegative
- Households cannot borrow infinitely until the end of their economic life cycle
- The households’ debt cannot increase at a rate asymptotically higher than the interest rate
The Ramsey Growth Model

The Household Problem

- Writing the Hamiltonian

\[ H = \int_{0}^{\infty} u(c_t) e^{(n-\rho)t} dt + \mu_t (w_t + (r_t - n) b_t - c_t) \]

FOCs with respect to \( c_t \) (control variable) and \( b_t \) (state variable).

\[ \frac{\partial H}{\partial c} = 0 : u'(c_t) e^{(n-\rho)t} = \mu_t \]

\[ \frac{\partial H}{\partial b} = -\dot{\mu} : (r_t - n) \mu_t = -\dot{\mu} \]

\[ \frac{\partial H}{\partial \mu} = \dot{b} : w_t + (r_t - n) \mu_t - c_t = \dot{b} \text{ (is the constraint)} \]
The Ramsey Growth Model

The Household Problem

- From the first FOC
  \[ u'(c_t) e^{(n-\rho)t} = \mu_t \implies \dot{\mu}_t = u''(c_t) e^{(n-\rho)t} \dot{c} + (n-\rho) u'(c_t) e^{(n-\rho)t} \]

- Combining with the second FOC, i.e. with \((r_t - n) \mu_t = -\dot{\mu}\), we get
  \[ (r_t - n) \mu_t = u''(c_t) e^{(n-\rho)t} \dot{c} + (n-\rho) u'(c_t) e^{(n-\rho)t} \]

or

\[ (r_t - n) u'(c_t) e^{(n-\rho)t} = u''(c_t) e^{(n-\rho)t} \dot{c} + (n-\rho) u'(c_t) e^{(n-\rho)t} \]

- Simplifying and rearranging we find the Euler equation
  \[ r_t = \rho - \left( \frac{u''(c) c}{u'(c)} \right) \frac{\dot{c}}{c} \]
The Ramsey Growth Model

The Household Problem

- Suppose that the utility is $u(c) = \frac{c^{1-\theta}}{1-\theta}$. \[ \implies -\frac{u''(c)c}{u'(c)} = \theta \] and the Euler equation becomes:

$$\frac{\dot{c}}{c} = \frac{1}{\theta} (r_t - \rho)$$

- The higher $r$, the more willing households are to save and shift consumption in the future.
- The higher the rate of return to consumption is, the more willing households are to sacrifice future consumption for more current consumption and thereby less current saving.
The Ramsey Growth Model

Firms Problem

- Profits Maximization

\[ \pi = F(K, L) - (r + \delta)K - wL \]

with \( F(K, L) \) Cobb-Douglas with constant return to scale

- Focs of the static problem

\[ f'(k) = (r + d) \]
\[ f(k) - kf'(k) = w \]
The Ramsey Growth Model

**Equilibrium**

- The law of motion of per capita capital
  \[ \dot{k} = f(k) - (n + d)k - c \]

- The law of motion of per capita consumption
  \[ \frac{\dot{c}}{c} = \frac{1}{\theta} \left( f'(k) - d - \rho \right) \]

- It is a $2 \times 2$ system of differential equation which defines the equilibrium.
The Ramsey Growth Model

The Steady State

\[ \dot{k} = 0 : f(k) - c = (n + d)k \]

The law of motion of per capita consumption

\[ \frac{\dot{c}}{c} = 0 : f'(k) = d + \rho \]
The Ramsey Growth Model

The Dynamics

- Notice that $\frac{\partial k}{\partial c} = -1 < 0$, while $\frac{\partial c}{\partial k} = \frac{1}{\theta} f''(k) < 0$
The Ramsey Growth Model

The Dynamics

- Remember that $c$ is a jumping variable while $k$ is a predetermined variable.
The Ramsey Growth Model

- **Variation in the rate of time preferences** $\rho$: a decrease in $\rho$ denotes that households have become less impatient and that they consider the value of future consumption higher. Thus, they are willing to sacrifice present consumption.

- For a given level of $k$, a decrease in $\rho$ lowers the consumption growth rate $\dot{c}$, and hence, shifts the $\dot{c} = 0$ schedule to the right.
The Ramsey Growth Model

- To maintain consumption constant the capital stock must increase so that, in the new long-run equilibrium, consumption, capital and output per capita attain a higher level ($\bar{k} > \bar{k}$, $\bar{c} > \bar{c}$).
- Consumption jumps along the saddle path, whereas $k$ at the same level $\bar{k}$. The reduction in consumption boosts saving and thereby leads to an increase in capital stock, until it reaches $\bar{k}$. $\Rightarrow$ income per capita level rises, with positive consumption growth rate until reaching the higher levels of consumption and capital ($\bar{c}$, $\bar{k}$), and income.
The Ramsey Growth Model

The Modified Golden Rule

- In the Solow model the Golden rule implies
  \[ f'(k^*) = n + d \]

- In the Ramsey model, the modified gold rule, which is obtained from consumer problem implies
  \[ f'(k^{Ramsey}) = \rho + d \]

- Given that the transersality condition implies that \( \rho > n \).

\[ \downarrow \]

- In the Ramsey model, the long-run equilibrium level of capital per worker is lower that in the Solow model. In the Ramsey model households save less since there is discounting of future utility.
The Ramsey Growth Model

- Is the decentralized equilibrium of the economy also the first best-outcome (the Pareto efficient equilibrium)?
- The social planner aims at maximizing households’ intertemporal utility subject to the aggregate resource constraint, which shows how the GDP is allocated to different uses.
- In a closed economy with no government, the aggregate output $Y$ is used either for consumption $C$ or for investment in physical capital, $I$, thus: $Y = C + I$
- Using that $I = \dot{K} + dK$

\[ Y = F(K, L) = C + \dot{K} + dK \]

- In terms of variables per worker

\[ f(k) = c + \dot{k} + (d + n) k \]
The Ramsey Growth Model

- The Social Planner Problem is

$$\max_{\{c,k\}} \int_0^\infty \frac{c^{1-\theta}}{1-\theta} e^{(n-\rho)t} dt$$

s.t. $$f(k) = c + \dot{k} + (d + n) k$$

and the Hamiltonian associated is

$$H = \frac{c^{1-\theta}}{1-\theta} e^{(n-\rho)t} - \mu [c + (d + n) k - f(k)]$$

FOCs wrt. consumption and capital

$$\frac{\partial H}{\partial c} = 0 : c^{-\theta} e^{(n-\rho)t} - \mu = 0$$

$$\frac{\partial H}{\partial k} = -\dot{\mu} : -\mu (d + n) - \dot{\mu} = 0$$
The Ramsey Growth Model

- combining the FOC

\[ \frac{\dot{c}}{c} = \frac{1}{\theta} \left[ f'(k) - (\rho + n) \right] \]

+ the low motion of capital accumulation

\[ \dot{k} = f(k) - c - (d + n) k \]

- in the steady state \( \dot{c} = \dot{k} = 0 \) and

\[ c = f(k) - (d + n) k \]
\[ f'(k) = \rho + n \]

- the Social Planner solution coincides with the decentralized solution.

\( \implies \) The competitive equilibrium in the Ramsey model is Pareto optimal.
The Solow Model with Human Capital

Mankiw, Romer and Weil (MRW 1992), "A Contribution to the Empirics of Economic Growth".

- They evaluate the performance of the Solow model
- They claim that the fit of the model can be improved by extending the model to include human capital $H$
- **We consider a simplified version of MRW model.**
- The Cobb-Douglas production function becomes

$$Y = K^{\alpha} (AH)^{1-\alpha}$$

- $A$ is the labor augmenting technology, which grows exogenously at a rate $g$ and $H$ is human capital, described by

$$H = e^{\psi u} L,$$

where $\psi$ is a positive constant and $u$ is the fraction of individual’s time spent learning skills. $L$ denotes the total amount of labor used in production in the economy.
The Solow Model with Human Capital

- Notice that if $u = 0$ then $H = L$, that is all labor is unskilled.
- By taking the first the log and then the derivative of $H$ with respect to $u$
  
  $$\log (H) = \psi u + \log (L)$$

  then

  $$\frac{d \log (H)}{du} = \psi \implies \frac{dH}{du} = \psi H$$

- Example $u = 1$ (one additional year of schooling), with $\psi = 0.1$ then $H$ increases by 10%.
- This formulation is intended to match the empirical literature on labor economics that finds that an additional year of schooling increases the productivity of the worker and than its wage by 10%.
Physical capital accumulates according to

\[ \dot{K} = s_k Y - dK \]

where \( s_k \) is the investment rate in physical capital and \( d \) is the depreciation rate.

The production function in terms of output per worker is

\[ y = k^\alpha (Ah)^{1-\alpha} \]

where \( h = e^{\psi u} \) is a constant.

We can redefine the production function in terms of \( y/(Ah) = \tilde{y} \)

\[ \tilde{y} = \tilde{k}^\alpha \]

and the capital accumulation becomes

\[ \dot{\tilde{k}} = s_k \tilde{y} - (n + g + d) \tilde{k} \]
The Solow Model with Human Capital

THE STEADY STATE

- \( \ddot{k} = 0 \) implies
  \[
  \frac{\ddot{k}}{\ddot{y}} = \frac{s_k}{(n + g + d)}
  \]

- substituting in the production function
  \[
  \ddot{y}^* = \left( \frac{s_k}{(n + g + d)} \right)^{\frac{\alpha}{1-\alpha}}
  \]

- in terms of output per worker
  \[
  y^*(t) = \left( \frac{s_k}{n + g + d} \right)^{\frac{\alpha}{1-\alpha}} hA(t)
  \]

- where \( t \) is included just to remind that \( A \) is an exogenous growing variable.
The Solow Model with Human Capital

From

\[ y^*(t) = \left( \frac{s_k}{n + g + d} \right)^{\frac{\alpha}{1-\alpha}} hA(t) \]

we can state that some countries are rich because they:

1. have higher investment in physical capital;
2. spend a large fraction of time accumulating skills \( h = e^{\psi u}; \)
3. have low population rate;
4. have higher level of technology.
To see how well the model performs empirically in explaining why some countries are rich and others are poor, we can consider relative income with respect to the US economy

\[
\hat{y}^* = \frac{y^*}{y_{US}}
\]

or

\[
\hat{y}^* = \left( \frac{\hat{s}_k}{\hat{x}} \right)^{\frac{\alpha}{1-\alpha}} \hat{h}\hat{A}
\]

where \( x = (n + g + d) \).
The Solow Model with Human Capital

- The Fit of the Neoclassical Growth model, 1997 (source: Jones book)
The Solow Model with Human Capital

- The lack of convergence for the world.
- Conditional convergence vs absolute convergence
The Green Solow Model and the Environmental Kunetz Curve

- Brock and Taylor (BT), 2010, Journal of Economic Growth
- Environmental Kunetz Curve (EKC): is a humpshaped relationship between environmental degradation and per capita income. At low level of economic activity, the environment is worsening. As the economic activity increases environmental degradation peacks. Then, as a country becomes richer and richer, environmental degradation begins to fall.
- BT use a simple variant of the basic Solow model with DR in capital and technologycal progress for abatement. They provide a theoretical explanation of the EKC and for abatement and emission intensity. They also derive an estimating equation for pollution convergence.
The Green Solow Model and the Environmental Kunetz Curve

US evidence:

1. Emissions per unit of output have been falling for a long period of time
US evidence:

2. Emission per unit of output were falling before absolute emissions started to fall
US evidence:

3. Abatement cost per unit of output have been relatively constant
The Green Solow Model and the Environmental Kunetz Curve

**EU evidence:**
- European data show a strong evidence of an Environmental Kunetz curve relationship
- The evidence on cross-country data is mixed
The Green Solow Model

- The Green Solow model augments the standard Solow model with pollution and abatement activity.
- The production function is

\[ Y = F(K, BL) \]

where \( B \) is the labor augmenting technology.
- The law of motion of capital is standard

\[ \dot{K} = sY - dK \]

- Net Emission of Pollution \( E \)

\[ E = \Omega F - \Omega A(F, F^A) = \Omega F \left( 1 - A\left(1, \frac{F^A}{F}\right) \right) = \Omega Fa(\theta) \]

where \( \Omega \) is pollution from \( Y \), \( A \) is the abatement with CRS production function \( A\left(1, \frac{F^A}{F}\right) \), and technology growth at exogenous rate \( g_A \). \( F \) is total production activity, \( F^A \) is total abatement activity. \( \theta = \frac{F^A}{F} \) and \( a(\theta) = 1 - A\left(1, \theta\right) \) with \( a(0) = 1 \), \( a'(\theta) < 0 \), and \( a''(\theta) > 0 \).
The Green Solow Model

- Output available for consumption and investment is
  \[ Y = F - F^A = (1 - \theta) F \]

- The model can be written in terms of efficient units of labor (EUL) as follows:

  \[
  \begin{align*}
  y &= f(k)(1 - \theta) \\
  \dot{k} &= sy - (n + d + g_B)k \\
  e &= f(k)\Omega a(\theta) 
  \end{align*}
  \]

Where \( k = K / BL \), \( y = Y / BL \) and \( e = E / BL \).
The Green Solow Model

Notice that \( a(\theta) = \text{constant} \). Given

\[
e = \frac{E}{BL} = f(k) \Omega a(\theta)
\]

we can rewrite

\[
E = BLf(k) \Omega a(\theta) = BLk^\alpha \Omega a(\theta)
\]

and thus, taking the growth rates

\[
\frac{\dot{E}}{E} = \frac{\dot{B}}{B} + \frac{\dot{L}}{L} + \alpha \frac{\dot{k}}{k} + \frac{\dot{\Omega}}{\Omega}
\]
The Green Solow Model

Easy to understand if we rewrite

\[ \frac{\dot{E}}{E} = \alpha \frac{\dot{k}}{k} + \left( g_B + n - g_A \right) \]

- **transitional component**
- **Emission growth along the BGP**
The Green Solow Model

- The growth rate of emission along the balanced growth path, i.e. along \( \frac{\dot{k}}{k} = 0 \) is
  \[ g_E = g_B + n - g_A \]

- We can define sustainable growth as the condition implying \( g_E < 0 \)
  
  1) \( g_B > 0 \)
  2) \( g_A > g_B + n \)

- 1) \( g_B > 0 \) is needed in order to have a positive growth of per capita income; 2) the technological progress in emission abatement has to be greater than the rate of growth of per capita GDP because of growth of population
The Green Solow Model

Considering

\[ \frac{\dot{E}}{E} = \alpha \frac{\dot{k}}{k} + g_B + n - g_A \]

and

\[ \frac{\dot{k}}{k} = s k^{\alpha - 1} (1 - \theta) - (d + n + g_B) \]

Assuming sustainable growth, i.e. \( g_E < 0 \) and multiplying by \( \alpha \) the second equation

\[ \alpha \frac{\dot{k}}{k} = \alpha s k^{\alpha - 1} (1 - \theta) - \alpha (d + n + g_B) \]
The Green Solow Model and the Environmental Kunetz Curve
The Green Solow Model and the Environmental Kunetz Curve

**PROPOSITION 1**

- If growth is sustainable $g_E < 0$ and $k(0) < k(T)$, then emissions grow initially and then fall continuously. An EKC occurs.
- If growth is sustainable $g_E < 0$ and $k(0) > k(T)$, then emissions fall continuously.
- If growth is unsustainable $g_E > 0$, then emissions growth but at a decreasing rate.

**PROPOSITION 2**

Identical economies with different initial values produce different per capita income and emission profiles over time. The peak level of emissions and the associated level of per capita income are not unique. This can explain the mixed evidence on the EKC in cross-country data. It is thus important to control for initial conditions and unobserved heterogeneity.
### Table 2  Carbon convergence across the world

<table>
<thead>
<tr>
<th>Variable</th>
<th>(A)</th>
<th>(B)</th>
<th>(C)</th>
<th>(D)</th>
<th>(E)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cons</td>
<td>0.021</td>
<td>-0.105</td>
<td>0.005</td>
<td>-0.005</td>
<td>-0.049</td>
</tr>
<tr>
<td></td>
<td>(11.6)</td>
<td>(-2.0)</td>
<td>(0.13)</td>
<td>(-0.12)</td>
<td>(-0.87)</td>
</tr>
<tr>
<td>log $e_t^C$</td>
<td>-0.005</td>
<td>-0.007</td>
<td>-0.008</td>
<td>-0.011</td>
<td>-0.012</td>
</tr>
<tr>
<td></td>
<td>(-5.6)</td>
<td>(-5.8)</td>
<td>(-5.9)</td>
<td>(-6.6)</td>
<td>(-6.7)</td>
</tr>
<tr>
<td>log $s$</td>
<td>-</td>
<td>0.029</td>
<td>0.023</td>
<td>0.035</td>
<td>0.041</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(3.4)</td>
<td>(2.5)</td>
<td>(3.9)</td>
<td>(4.3)</td>
</tr>
<tr>
<td>log($n + g_B + \delta$)</td>
<td>-</td>
<td>0.019</td>
<td>-0.029</td>
<td>-0.045</td>
<td>-0.031</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.93)</td>
<td>(-1.9)</td>
<td>(-2.4)</td>
<td>(-1.35)</td>
</tr>
<tr>
<td>$N$</td>
<td>173</td>
<td>165</td>
<td>151</td>
<td>118</td>
<td>98</td>
</tr>
<tr>
<td>Adj. $R^2$</td>
<td>0.14</td>
<td>0.29</td>
<td>0.31</td>
<td>0.35</td>
<td>0.42</td>
</tr>
</tbody>
</table>

*Notes*  $t$-statistics are in parentheses. Column A estimates the short version of our model using Sample A described in the text, while the remaining columns estimate a version of our long specification using Samples B–E. The long version is: $[1/N] \log[e_{it}^C/e_{it-N}^C] = \beta_0 + \beta_1 \log[e_{it-N}^C] + \beta_2 \log[s_i] + \beta_3 \log[(n + g_B + \delta)_i] + \mu_{it}$, where the dependent variable is the average growth rate in log emissions per capita over the 1998–1960 period, $e_{it-N}^C$ is emissions per capita in 1960, $s$ is the average investment to GDP ratio over the 1960–1998 period, and $(n + g_B + \delta)$ is average population growth over the period plus 0.05 (and therefore captures only variation in population growth rates across countries).
The econometric results reported by Brock and Taylor (2010) in Table 2 show that

1. the coefficient on $\ln(e^C)$ is negative and statistically significant across countries. Thus conditional convergence in per capita emissions emerges.
2. The coefficient on $\ln(s)$ is positive and the coefficient on $\ln(n + g + d)$ is negative, as predicted by the green Solow model.