

Algebra Notes on Blanchard and Kiyotaki (1987).

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## I. Basic Setup

There are  $i = 1, \dots, m$  firms which have the following technology,

$$Y_i = \left( \sum_{j=1}^n N_{i,j}^{(\sigma-1)/\sigma} \right)^{\sigma/(\sigma-1)(1/\alpha)} \quad (1)$$

where  $y_i$  is the output of firm  $i$ ,  $N_{i,j}$  is the quantity of labor of type  $j$  used in production of output  $i$ . There are  $n$  types of labor index  $j = 1, \dots, n$  and  $\alpha - 1$  is the elasticity of marginal cost with respect to output. We impose that  $\sigma > 1$  and  $\alpha \geq 1$ . Firms maximize profits,

$$V_i = P_i y_i - \sum_{j=1}^n w_j N_{i,j} \quad (2)$$

where  $P_i$  is the nominal output price of firm  $i$  and  $w_j$  is the nominal wage of labor type  $j$  subject to (1). The  $j = 1, \dots, n$  households supply labor, hold money and consume goods. The utility function is written as,

$$U_j = \left[ m^{1/(1-\theta)} \left( \sum_{i=1}^m C_{i,j}^{(\theta-1)/\theta} \right)^{\theta/(\theta-1)} \right]^\gamma (M'_j/P)^{1-\gamma} - N_j^\beta \quad (3)$$

where,

$$C_j = m^{1/(1-\theta)} \left( \sum_{i=1}^m C_{i,j}^{(\theta-1)/\theta} \right)^{\theta/(\theta-1)} \quad (4)$$

is the consumption of good  $i$  by household  $j$ .<sup>1</sup>

$$P = \left( \sum_{i=1}^m P_i^{1-\theta} \right)^{1/(1-\theta)} \quad (5)$$

is a price index. The  $m^{1/(1-\theta)}$  term is a normalization with the implication that an increase in the number of products does not affect the marginal utility after optimization.

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<sup>1</sup>Note that the consumption index is defined differently from the one in Blanchard and Kiyotaki (1987), however, the utility function is the same.

The term  $\theta$  is the elasticity of substitution between goods, and it is restricted to be greater than one. Finally,  $\beta - 1$  is the elasticity of the marginal dis-utility of leisure.<sup>2</sup> The household face a budget constraint of the form,

$$\sum_{i=1}^m P_i C_{i,j} + M'_j = w_j N_{i,j} + M_j + \sum_{i=1}^m V_{i,j} \equiv I_j \quad (6)$$

where  $M_j$  is the initial endowment of money and  $V_{i,j}$  the share of profits of firm  $i$  going to household  $j$ .

Equilibrium is characterized as a relation between real money balances and aggregate demand, a pair of demand functions for goods and labor and a pair of price and wage rules.

## II. Demands for Product and Labor Types

Here I consider the household and firm problems.

### *Iia. Demand for Product of type $i$*

Agents pick  $\{C_{i,j}\}_{i=1}^m, M'_j, w_j, N_j$  to maximize (3) subject to (6). Because utility is separable in labor for now I consider the problem,

$$\max_{\{C_{i,j}\}_{i=1}^m, M'_j} U_j = B_j^\gamma (M'_j)^{1-\gamma} / P \quad (7)$$

where,

$$B_j \equiv PC_j = \sum_{i=1}^m P_i C_{i,j}$$

$$C_j \equiv m^{1/(1-\theta)} \left( \sum_{i=1}^m C_{i,j}^{(\theta-1)/\theta} \right)^{\theta/(\theta-1)}$$

so that  $B_j + M'_j = I_j$ . Now, take the natural logarithm of (7) and take the derivative. This implies,

$$\partial U_j / \partial B_j = \gamma / B_j + (1 - \gamma) / (B_j - I_j) = 0$$

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<sup>2</sup>Let  $D \equiv -\beta N^{B-1}$ . Then,  $(\partial D / \partial N) N / D = -\beta(\beta - 1) N^{\beta-2} (N / (-\beta N^{B-1})) = \beta - 1$  is the elasticity.

so that we arrive at the condition,

$$M'_j = I_j - B_j = (1 - \gamma) I_j \quad (8)$$

which is given by equation A2 in BK. Now consider the indirect utility function. Of course we also have,

$$B_j = PC_j = \gamma I_j$$

Recall that the indirect utility function from standard consumer theory is simple the optimal condition substituted into the direct utility function, here given by (7). Making the required substitution now implies,

$$\Lambda_j = \gamma I_j^\gamma ((1 - \gamma) I_j)^{1-\gamma} / P \equiv \mu I_j / P \quad (9)$$

where I define  $\mu \equiv (1 - \gamma)^{(1-\gamma)} \gamma^\gamma$ . Equation (9) is given by A3 in BK.

To find equation A1 in BK we first need to solve another problem. This is basically a two-stage budgeting problem. I write,

$$\max_{\{C_{i,j}\}_{i=1}^m} \mathcal{L} = C_j - \lambda \left( \sum_{i=1}^m P_i C_{i,j} - B_j \right)$$

which implies I need  $\partial \mathcal{L} / \partial C_{i,j} = 0$  for an optimum. But this expression also implies  $\partial C_j / \partial C_{i,j} = \lambda P_i$ , or rather,

$$\partial C_j / \partial C_{i,j} = m^{1/(1-\theta)} \left( \frac{\theta}{\theta - 1} \right) \left( \sum_{i=1}^m C_{i,j}^{(\theta-1)/\theta} \right)^{(\theta/(\theta-1))-1} \left( \frac{\theta - 1}{\theta} \right) C_{i,j}^{((\theta-1)/\theta)-1}$$

by a simple application of the chain rule. It is possible to simplify this to,

$$\partial C_j / \partial C_{i,j} = m^{1/(1-\theta)} \left( \sum_{i=1}^m C_{i,j}^{(\theta-1)/\theta} \right)^{1/(\theta-1)} C_{i,j}^{-1/\theta}$$

or,

$$\begin{aligned} \lambda P_i &= m^{1/(1-\theta)} (C_j / m^{1/(1-\theta)})^{1/\theta} C_{i,j}^{-1/\theta} \\ (\lambda P_i)^\theta &= m^{\theta/(1-\theta)} (C_j / m^{1/(1-\theta)}) (1/C_{i,j}) \\ (\lambda P_i)^\theta &= (C_j / m C_{i,j}) \\ C_{i,j} &= (\lambda P_i)^{-\theta} (\gamma I_j / m P) \end{aligned} \quad (10)$$

where in the last line we have used the fact that  $C_j = \gamma I_j / P$ .

Now all we need to find is  $\lambda$ . To get it we use Euler's theorem for homogenous functions ( $C_j$  is homogenous of degree 1). In this case from the derivative of the Lagrangian,

$$\begin{aligned} (\partial C_j / \partial C_{i,j}) C_{i,j} &= \lambda P_i C_{i,j} \\ \sum_{i=1}^m (\partial C_j / \partial C_{i,j}) C_{i,j} &= \lambda \sum_{i=1}^m P_i C_{i,j} \end{aligned}$$

which can be re-expressed as,

$$C_j = \lambda B_j = \lambda P C_j$$

so that  $\lambda = 1/P$ . Using this condition in (10) we arrive at,

$$C_{i,j} = (P_i/P)^{-\theta} (\gamma I_j / mP) \quad (11)$$

and this is given by A1 in BK. Now I need to deal with the price index. Using (11) and the previous definitions note I can write,

$$P C_j = B_j = \sum_{i=1}^m P_i C_{i,j} = \sum_{i=1}^m P_i (P_i/P)^{-\theta} (\gamma I_j / mP)$$

Using (8) I re-express this as,

$$\begin{aligned} P C_j &= \sum_{i=1}^m P_i (P_i/P)^{-\theta} (c_j / m) \\ P &= (1/P)^{-\theta} \sum_{i=1}^m P_i^{1-\theta} (1/m) \\ P &= \left( (1/m) \sum_{i=1}^m P_i^{1-\theta} \right)^{1/(1-\theta)} \end{aligned} \quad (12)$$

which is equation (A4) in BK. Given this, it is now possible to find an expression for aggregate demand. First note that I have,

$$Y_i = \sum_{j=1}^n C_{i,j} = \gamma (P_i/P)^{-\theta} \left( \sum_{j=1}^n I_j \right) / mP$$

It is also true that,

$$\begin{aligned} Y &= \left( \sum_{i=1}^m P_i y_i \right) / P = \left( \sum_{i=1}^m P_i \sum_{j=1}^n C_{i,j} \right) / P = \left( \sum_{i=1}^m \sum_{j=1}^n P_i C_{i,j} \right) / P \\ &= \left( \sum_{j=1}^n B_j \right) / P = \gamma \left( \sum_{j=1}^n I_j \right) / P \end{aligned} \quad (13)$$

which is (A6) in BK. Given this we have an expression for the demand for product  $i$

$$Y_i = (P_i/P)^{-\theta} Y/m \quad (14)$$

which is (A5) in BK. Finally, note that (11), (8), (14) and (13) together imply,

$$Y = (\gamma/(1-\gamma)) M'/P \quad (15)$$

where  $M' \equiv \sum_{j=1}^m M'_j$ . This is given by (A7) in BK.

### *Iib. Demand for Labor of type $j$*

Let's define the following labour index that has the same structure of the consumption index used before,

$$L_i = n^{1/(1-\sigma)} \left( \sum_{j=1}^n N_{i,j}^{(\sigma-1)/\sigma} \right)^{\sigma/(\sigma-1)}$$

Thus, the production function becomes,

$$Y_i = (n^{1/(\sigma-1)} L_i)^{1/\alpha}$$

and then,

$$L_i = n^{1/(1-\sigma)} Y_i^\alpha$$

The problem for the firm is to pick  $\{N_{i,j}\}_{j=1}^n$  to minimize  $\sum_{j=1}^n w_j N_{i,j}$  subject to,

$$n^{1/(1-\sigma)} \left( \sum_{j=1}^n N_{i,j}^{(\sigma-1)/\sigma} \right)^{\sigma/(\sigma-1)} = \bar{L}_i$$

Write the Lagrangian as,

$$\mathcal{L} = \sum_{j=1}^n w_j N_{i,j} - \rho \left( n^{1/(1-\sigma)} \left( \sum_{j=1}^n N_{i,j}^{(\sigma-1)/\sigma} \right)^{\sigma/(\sigma-1)} - \bar{L}_i \right)$$

Clearly, we want  $\partial\mathcal{L}/\partial N_{i,j} = 0$  or  $w_j = \rho(\partial L_i/\partial N_{i,j})$ . From the definition of  $L_i$  I can re-express this as,

$$\begin{aligned}
(\partial L_i/\partial N_{i,j}) &= n^{1/(1-\sigma)} \frac{\sigma}{\sigma-1} \left( \sum_{j=1}^n N_{i,j}^{(\sigma-1)/\sigma} \right)^{(\sigma/(\sigma-1))-1} \frac{\sigma-1}{\sigma} N_{i,j}^{((\sigma-1)/\sigma)-1} \\
&= n^{1/(1-\sigma)} \left( \sum_{j=1}^n N_{i,j}^{(\sigma-1)/\sigma} \right)^{(1/(\sigma-1))} N_{i,j}^{-1/\sigma} \\
&= n^{1/(1-\sigma)} \left( \left( \sum_{j=1}^n N_{i,j}^{(\sigma-1)/\sigma} \right)^{(\sigma/(\sigma-1))} \right)^{1/\sigma} N_{i,j}^{-1/\sigma} \\
&= n^{1/(1-\sigma)} (L_i/n^{1/(1-\sigma)})^{1/\sigma} N_{i,j}^{-1/\sigma}
\end{aligned}$$

and therefore,

$$\begin{aligned}
(w_j/\rho)^\sigma &= n^{\sigma/(1-\sigma)} (L_i/n^{1/(1-\sigma)}) N_{i,j} \\
(w_j/\rho)^\sigma &= n^{-1} L_i N_{i,j} \\
N_{i,j} &= n^{-1} L_i / (w_j/\rho)^{-\sigma}
\end{aligned}$$

As before it is necessary to find  $\rho$  but again we can exploit Euler's theorem and we find analogously that  $\rho = w$ . We also have,

$$w = \left( (1/n) \sum_{j=1}^n w_j^{1-\sigma} \right)^{1/(1-\sigma)} \tag{16}$$

where,

$$wL_i = \sum_{j=1}^n w_j N_{i,j} \tag{17}$$

which are A9 and A8 respectively in BK. Finally we can also simply derive,

$$\begin{aligned}
N_{i,j} &= (w_j/w)^{-\sigma} L_i n^{-1} \\
&= (w_j/w)^{-\sigma} (n^{1/(1-\sigma)}/n) Y_i^\alpha \\
&= (w_j/w)^{-\sigma} n^{\sigma/(1-\sigma)} Y_i^\alpha
\end{aligned} \tag{18}$$

Aggregating these expression we find

$$\begin{aligned}
N &= \sum_{j=1}^n w_j N_j / w = \sum_{j=1}^n w_j N_j \sum_{i=1}^m N_{i,j} / w = \sum_{j=1}^n \sum_{i=1}^m w_j N_{i,j} / w \\
&= \sum_{i=1}^m w L_i / w = \sum_{i=1}^m L_i = n^{1/(1-\sigma)} \sum_{i=1}^m Y_i^\alpha
\end{aligned} \tag{19}$$

which is A11 in BK.

Furthermore, we obtain the demand for labour of type  $j$

$$N_j = \sum_{i=1}^m N_{i,j} = (w_j/w)^{-\sigma} n^{-1} \sum_{i=1}^m L_i = (w_j/w)^{-\sigma} (N/n)$$

that is equation A10 in BK.

### III. Price and Wage Rules

Recall equation (2),  $V_i = P_i Y_i - \sum_{j=1}^n w_j N_{i,j}$ , this is given as A12 in BK. The problem is to maximize profits,  $V_i$ , subject to (14) and (17). First note that this allows us to write,

$$\max V_i = P_i(Y_i) \cdot Y_i - w n^{\sigma/(1-\sigma)} Y_i^\alpha$$

where is the inverse demand function. Taking the derivative with respect to  $P_i$  I find,

$$\partial V_i / \partial Y_i = P_i + Y_i (dP_i / dY_i) - \alpha w n^{\sigma/(1-\sigma)} Y_i^{\alpha-1}$$

where the first term on the RHS is the marginal revenue and the second the marginal cost. Importantly we can re-express this (setting  $\partial V_i / \partial Y_i = 0$ ) as,

$$\begin{aligned}
MC &= P_i (1 + (Y_i/P_i) (dP_i/dY_i)) \\
&= P_i (1 + ((Y_i/P_i) (dP_i/dY_i))^{-1}) \\
&= P_i (1 - \theta^{-1})
\end{aligned}$$

so that,

$$P_i = (\theta / (\theta - 1)) MC \tag{20}$$

where  $\theta$  is the price elasticity of output and  $\theta/(\theta - 1)$  is interpreted as the monopoly markup. Returning to the expression for marginal cost,

$$MC = \alpha w n^{\sigma/(1-\sigma)} \left( (P_i/P)^{-\theta} Y/m \right)^{\alpha-1}$$

by equation (14). Combing this with (6) then I can write,

$$P_i (\theta - 1) / \theta = \alpha w n^{\sigma/(1-\sigma)} \left( (P_i/P)^{-\theta} Y/m \right)^{\alpha-1}$$

or,

$$P_i/P = \left( (\theta/(\theta - 1)) (w/P) n^{\sigma/(1-\sigma)} \alpha (Y/m)^{\alpha-1} \right)^{1/(1+\theta(\alpha-1))} \quad (21)$$

This is the price rule and is given by A13 in BK. Now I need to get the wage rule. Consider the following,

$$\max_{\{N_j\}} U_j = \mu I_j/P - N_j^\beta \quad (22)$$

subject to,

$$w_j N_j + M_j + \sum_{i=1}^m V_{i,j} = I_j \quad (23)$$

$$N_j = \sum_{i=1}^m N_{i,j} = (w_j/w)^{-\sigma} n^{-1} \sum_{i=1}^m L_i = (w_j/w)^{-\sigma} (N/n) \quad (24)$$

where (22) and (23) are given by A14 and A15 in BK. Equivalently then,

$$\max_{\{N_j\}_{j=1}^n} U_j = \mu \left\{ \frac{w_j N_j + M_j + \sum_{i=1}^m V_{i,j}}{P} \right\} - N_j^\beta$$

Similarly this produces a simply relation between marginal revenue and marginal cost,

$$\partial U_j / \partial N_j = (\mu/P) \{w_j + N_j (dw_j/dN_j)\} - \beta N_j^{\beta-1}$$

As before , once the elasticities are defined, I can write,

$$\begin{aligned} MR &= (\mu/P) \{w_j + N_j (dw_j/dN_j)\} \\ &= (\mu/P) w_j \{1 + (N_j/w_j) (dw_j/dN_j)\} \\ &= (\mu/P) w_j \{1 - 1/\sigma\} \end{aligned}$$

or,

$$(\mu/P) w_j = (\sigma/(\sigma - 1)) MC$$

I also have,

$$MC = \beta N_j^{\beta-1} = \beta [(w_j/w)^{-\sigma} (N/n)]^{\beta-1} \quad (25)$$

so that the wage rule is,

$$w_j/w = \left( (\sigma/(\sigma - 1)) (P/w) (\beta/\mu) (N/n)^{\beta-1} \right)^{1/(1+\sigma(\beta-1))} \quad (26)$$

which is A16 in BK. Finally there are equations characterizing the money market. Aggregate demand implies,

$$M' = \sum_{j=1}^n M'_j = (1 - \gamma) \sum_{j=1}^n I_j = ((1 - \gamma)/\gamma) PY \quad (27)$$

in equilibrium we have that  $M' = M$ , thus A7 becomes

$$Y = ((1 - \gamma)/\gamma) M/P \quad (28)$$

which is A17 in BK.

#### IV. Symmetric Flex-price/Flex-wage Equilibrium

Now consider the flex-price/Flex-wage equilibrium. This is when  $P_i = P \forall i$  and  $w_j = w \forall j$ . In this case (21) and (26) reduce to the following pair of equations,

$$P/w = (\theta/(\theta - 1)) \underbrace{(n^{1/(1-\sigma)} \alpha (Y/m)^{\alpha-1})}_{\equiv \kappa^P}$$

$$w/P = (\sigma/(\sigma - 1)) \underbrace{(n^{\sigma/(1-\sigma)} m^{(1-\alpha)(\beta-1)} Y^{\alpha(\beta-1)})}_{\equiv \kappa^w}$$

Finally, I have,

$$N = \sum_{i=1}^m L_i = n^{1/(1-\sigma)} \sum_{i=1}^m Y_i^\alpha$$

$$= n^{1/(1-\sigma)} \sum_{i=1}^m \left( (P_i/P)^{-\theta} Y/m \right)^\alpha \quad (29)$$

$$= n^{1/(1-\sigma)} m^{1-\alpha} Y^\alpha \quad (30)$$

where (29) and (30) are given as A18 and A19 in BK.

## **V. References**

**Blanchard, O. J., Kiyotaki, N.**, 1987. Monopolistic Competition and the Effects of Aggregate Demand, *American Economic Review* 77, 647-666.