Seemingly Unrelated Regression Equations

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Preliminaries. Kronecker Product

$A \ (m \times n), \ B \ (p \times q)$

$$A \otimes B = \begin{bmatrix} a_{11}B & \ldots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & a_{mn}B \end{bmatrix} \ (mp \times nq)$$

• $A_1, A_2 \ (m \times n) \ B_1, B_2 \ (p \times q)$

$$D_i = (A_i \otimes B_1) \quad i = 1, 2$$

$$D_1 + D_2 = (A_1 + A_2) \otimes B_1$$

• $C = (A \otimes B)$

$$\alpha C = (\alpha A \otimes B) = (A \otimes \alpha B)$$
Preliminaries. Kronecker Product

- $C_i = (A_i \otimes B_i)$ $i = 1, 2$

  $C_1 C_2 = A_1 A_2 \otimes B_1 B_2$

- $C = (A \otimes B)$

  $C' = (A' \otimes B')$

- $C = (A \otimes B)$

  $C^{-1} = (A^{-1} \otimes B^{-1})$

- $tr (A \otimes B) = tr (A) tr (B)$
Structural Form:

$$\Gamma'y_t = B'x_t + \varepsilon_t$$

$$\Gamma'E (y_t | \mathcal{F}_t) = B'x_t$$

$$y_t \ (g \times 1)$$

$$x_t \ (k \times 1)$$

$$\mathcal{F}_t = \sigma (x_t) \lor \mathcal{Y}_{t-1} \lor \mathcal{X}_{t-1}$$

$$\mathcal{Y}_t \lor \mathcal{X}_t$$ $\sigma$-fields generated by the histories of $y_t$ and $x_t$ respectively.

$$\Gamma \ (g \times g)$$ invertible, structural coefficients.
Each row of $\mathbf{B}$ has one element set to one.

$\mathbf{B}$ ($k \times g$) structural coefficients.

The model is complete: number of equations = number of endogenous variables.

\[
E(\varepsilon_t | \mathcal{F}_t) = 0
\]

\[
E(\varepsilon_t \varepsilon_t' | \mathcal{F}_t) = \Sigma.
\]

\[
\Gamma' E(y_t | \mathcal{F}_t) = \mathbf{B}' x_t
\]

$x_t \in \mathcal{F}_t$

\[
E(y_t | \mathcal{F}_t) = \Gamma'^{-1} \mathbf{B}' x_t
\]
Simultaneous equations

Reduced Form

\[ E(\mathbf{y}_t | \mathcal{F}_t) = \Pi \mathbf{x}_t \]

Structural disturbance term

\[ \varepsilon_t = \Gamma' (\mathbf{y}_t - E(\mathbf{y}_t | \mathcal{F}_t)) = \Gamma' \mathbf{y}_t - \mathbf{B}' \mathbf{x}_t \]

Reduced form disturbance term

\[ \mathbf{v}_t = \Gamma'^{-1} \varepsilon_t = \mathbf{y}_t - \Pi \mathbf{x}_t \]

Particular case:

\[ \Gamma' = \mathbf{I}_G \]

\[ \mathbf{y}_t = \mathbf{B}' \mathbf{x}_t + \varepsilon_t \]

This is a system of *Seemingly Unrelated Regression Equations*.
The econometric specification is called seemingly unrelated regression equations (SURE) because the individual regression equations have no structural relationship in the sense that $y_t$ does not appear as an RHS variable in the $i$–th ($i \neq j$) equation.

The OLS estimator fails to take into account cross-equation information that can be exploited to improve estimator efficiency.
**Hypothesis** $m$ equations

\[ y_{1t} = x_{1t}'\beta_1 + \varepsilon_{1t} \]

\[ \vdots \]

\[ y_{mt} = x_{mt}'\beta_m + \varepsilon_{mt} \]

\[ x_{it} \quad (k_i \times 1) \quad i = 1, \ldots, m \]

\[ \beta_i \quad (k_i \times 1) \]

\[ E (\varepsilon_{it} | X_1, \ldots, X_m) = 0 \quad i = 1, \ldots, m \]

\[ E (\varepsilon_{it}^2 | X_1, \ldots, X_m) = \sigma_{ii} \quad i = 1, \ldots, m \]

\[ E (\varepsilon_{it}\varepsilon_{it-k} | X_1, \ldots, X_m) = 0 \quad k \neq 0 \quad i = 1, \ldots, m \]

\[ E (\varepsilon_{it}\varepsilon_{jt} | X_1, \ldots, X_m) = \sigma_{ij} \neq 0 \quad i, j = 1, \ldots, m \quad i \neq j \]

\[ E (\varepsilon_{it}\varepsilon_{jt-k} | X_1, \ldots, X_m) = 0 \quad k \neq 0 \]
Seemingly Unrelated Regression Equations

\[ \varepsilon_t = \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \vdots \\ \varepsilon_{mt} \end{bmatrix} \]

\( \varepsilon_t \) is a vector white noise

\[ E(\varepsilon_t \varepsilon'_t | X_1, \ldots, X_m) = \Sigma = \{\sigma_{ij}\} \quad i, j = 1, \ldots, m \]

\[ Y^1 = X_1 \beta_1 + \varepsilon^1 \]

\[ \vdots \]

\[ Y^m = X_m \beta_m + \varepsilon^m \]
Seemingly Unrelated Regression Equations

\[ \begin{align*}
Y_i & \quad (T \times 1) \\
X_i & \quad (T \times k_i) \quad i = 1, \ldots, m \\
\varepsilon^i & \quad (T \times 1)
\end{align*} \]

\[ 
\begin{align*}
E(\varepsilon^i | X_1, \ldots, X_m) &= 0 \\
E(\varepsilon^i \varepsilon^i' | X_1, \ldots, X_m) &= \sigma_{ii} I_T \\
E(\varepsilon^i \varepsilon^j' | X_1, \ldots, X_m) &= \sigma_{ij} I_T
\end{align*} \]
Seemingly Unrelated Regression Equations

\[
\begin{bmatrix}
Y^1 \\
\vdots \\
Y^m
\end{bmatrix} =
\begin{bmatrix}
X_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & X_m
\end{bmatrix}
\begin{bmatrix}
\beta_1 \\
\vdots \\
\beta_m
\end{bmatrix} +
\begin{bmatrix}
\varepsilon^1 \\
\vdots \\
\varepsilon^m
\end{bmatrix}
\]

\[Y = Z\delta + \varepsilon\]

where

\[
Z =
\begin{bmatrix}
X_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & X_m
\end{bmatrix}
\]
Seemingly Unrelated Regression Equations

\[ \delta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix} \]
Seemingly Unrelated Regression Equations

\[ E(\varepsilon\varepsilon' | Z) = \Sigma \otimes I_T \]

\[
E(\varepsilon\varepsilon' | Z) = \\
= \begin{bmatrix}
E(\varepsilon^1\varepsilon^{1'} | Z) & \cdots & E(\varepsilon^1\varepsilon^{m'} | Z) \\
\vdots & \ddots & \vdots \\
E(\varepsilon^m\varepsilon^{1'} | Z) & \cdots & E(\varepsilon^m\varepsilon^{m'} | Z)
\end{bmatrix}
= \begin{bmatrix}
\sigma_{11}I_T & \cdots & \sigma_{1m}I_T \\
\vdots & \ddots & \vdots \\
\sigma_{m1}I_T & \cdots & \sigma_{mm}I_T
\end{bmatrix}
\]
Seemingly Unrelated Regression Equations

Let

\[ \Sigma \otimes I_T = \Omega \]

the GLS estimator is:

\[ \hat{\beta}_{GLS} = (X'\Omega^{-1}X)^{-1} X'\Omega^{-1}Y \]

in this case:

\[ \hat{\delta}_{GLS} = (Z'\Omega^{-1}Z)^{-1} Z'\Omega^{-1}Y \]

\[ = \left( Z' (\Sigma \otimes I_T)^{-1} Z \right)^{-1} Z' (\Sigma \otimes I_T)^{-1} Y \]

\[ = \left( Z' (\Sigma^{-1} \otimes I_T) Z \right)^{-1} Z' (\Sigma^{-1} \otimes I_T) Y \]
Feasible GLS (Aitken-Zellner Estimator)

Estimate of $\Sigma$

$$\hat{\sigma}_{ij} = \frac{\hat{\varepsilon}_i^T \hat{\varepsilon}_j}{T}$$

$$\hat{\varepsilon}_i = Y^i - \hat{Y}^i = Y^i - X_i \hat{\beta}_{iOLS}$$

$$\hat{\delta}_{FGLS} = \left( Z' \left( \hat{\Sigma}^{-1} \otimes I_T \right) Z \right)^{-1} Z' \left( \hat{\Sigma}^{-1} \otimes I_T \right) Y$$
Two particular cases:

- $\Sigma$ diagonal. There is no relation among the equations.
- $X_i = X$.

Then $\hat{\delta}_{OLS} = \hat{\delta}_{GLS}$ and $Var\left(\hat{\delta}_{OLS} \mid Z\right) = Var\left(\hat{\delta}_{GLS} \mid Z\right)$. 
Σ diagonal Assume that $m = 2$.

\[ Y^1 = X_1 \beta_1 + \varepsilon^1 \]
\[ Y^2 = X_2 \beta_2 + \varepsilon^2 \]

\[
E (\varepsilon^1 \varepsilon^{1'} \mid X_1, X_1) = \sigma_{11} I_T \\
E (\varepsilon^2 \varepsilon^{2'} \mid X_1, X_1) = \sigma_{22} I_T \\
E (\varepsilon^1 \varepsilon^{2'} \mid X_1, X_1) = \sigma_{12} I_T = 0 \\
\sigma_{12} = 0
\]

\[
\Sigma^{-1} = \begin{bmatrix}
\sigma_{11}^{-1} & 0 \\
0 & \sigma_{22}^{-1}
\end{bmatrix}
\]
\[ \hat{\delta}_{GLS} = (Z' (\Sigma^{-1} \otimes I_T) Z)^{-1} Z' (\Sigma^{-1} \otimes I_T) Y \]

\[ \hat{\delta} = \left\{ \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}' \begin{bmatrix} \sigma_{11}^{-1} I_T & 0 \\ 0 & \sigma_{22}^{-1} I_T \end{bmatrix} \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \right\}^{-1} \times \left\{ \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}' \begin{bmatrix} \sigma_{11}^{-1} I_T & 0 \\ 0 & \sigma_{22}^{-1} I_T \end{bmatrix} \begin{bmatrix} Y^1 \\ Y^2 \end{bmatrix} \right\} \]
\[ \hat{\delta}_{GLS} = \left\{ \begin{bmatrix} \sigma^{-1}_{11}X'_1X_1 & 0 \\ 0 & \sigma^{-1}_{22}X'_2X_2 \end{bmatrix} \right\}^{-1} \begin{bmatrix} \sigma^{-1}_{11}X'_1 & 0 \\ 0 & \sigma^{-1}_{22}X'_2 \end{bmatrix} \begin{bmatrix} Y^1 \\ Y^2 \end{bmatrix} \]

\[ = \begin{bmatrix} (X'_1X_1)^{-1}X'_1Y^1 \\ (X'_2X_2)^{-1}X'_2Y^2 \end{bmatrix} = \hat{\delta}_{OLS}. \]
Every regression function contains the same explanatory variables, $X_i = X$.

$$Z = \begin{bmatrix}
X & 0 & \cdots & 0 \\
0 & X & \vdots \\
\vdots & \ddots & 0 \\
0 & \cdots & 0 & X
\end{bmatrix}$$

$$Z = I_m \otimes X$$
\[
\widehat{\delta}_{GLS} = (Z' (\Sigma^{-1} \otimes I_T) Z)^{-1} Z' (\Sigma^{-1} \otimes I_T) Y
\]

\[
= [(I_m \otimes X)' (\Sigma^{-1} \otimes I_T) (I_m \otimes X)]^{-1} (I_m \otimes X)' (\Sigma^{-1} \otimes I_T) Y
\]

\[
= [(\Sigma^{-1} \otimes X') (I_m \otimes X)]^{-1} (\Sigma^{-1} \otimes X') Y
\]

\[
= [(\Sigma^{-1} \otimes (X'X))]^{-1} (\Sigma^{-1} \otimes X') Y
\]

\[
= \left(\Sigma \otimes (X'X)^{-1}\right) (\Sigma^{-1} \otimes X') Y
\]

\[
= \left[I_m \otimes (X'X)^{-1} X'\right] Y
\]
\[ \hat{\delta}_{GLS} = \begin{pmatrix} (X'X)^{-1} X' & 0 & \cdots & 0 \\ 0 & (X'X)^{-1} X' & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (X'X)^{-1} X' \end{pmatrix} \begin{bmatrix} Y^1 \\ Y^2 \\ \vdots \\ Y^m \end{bmatrix} \]

\[ \hat{\delta}_{GLS} = \begin{bmatrix} (X'X)^{-1} X'Y^1 \\ (X'X)^{-1} X'Y^2 \\ \vdots \\ (X'X)^{-1} X'Y^2 \end{bmatrix} = \hat{\delta}_{OLS} \]

\[ V \text{ar} \left( \hat{\delta}_{GLS} \mid Z \right) = \left[ (I_m \otimes X)' (\Sigma^{-1} \otimes I_T) (I_m \otimes X) \right]^{-1} = \Sigma \otimes (X'X)^{-1} = V \text{ar} \left( \hat{\delta}_{OLS} \mid Z \right). \]