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Introduction to Time Series

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The value of y at date t depends on p of its own lags along with the current of the input variable (w_t):

$$y_t = \phi_1 y_{t-1} + \ldots + \phi_p y_{t-p} + w_t$$

Companion Form

$$\xi_t = \begin{bmatrix} y_t \\ \vdots \\ y_{t-p+1} \end{bmatrix} \quad (p \times 1) \quad \xi_{t-1} = \begin{bmatrix} y_{t-1} \\ \vdots \\ y_{t-p} \end{bmatrix} \quad (p \times 1)$$

$$F = \begin{bmatrix} \phi_1 & \phi_2 & \dots & \phi_{p-1} & \phi_p \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \quad (p \times p) \qquad v_t = \begin{bmatrix} w_t \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (p \times 1)$$

First-order difference equation

$$\xi_t = F\xi_{t-1} + v_t$$

It is a system of p equations

$$\begin{aligned} y_t &= \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + w_t \\ y_{t-1} &= y_{t-1} \\ \vdots &= \vdots \end{aligned}$$

$$\begin{aligned}\xi_0 &= F\xi_{-1} + v_0 \\ \xi_1 &= F\xi_0 + v_1 = F(F\xi_{-1} + v_0) + v_1 = F^2\xi_{-1} + Fv_0 + v_1\end{aligned}$$

Recursively,

$$\xi_t = F^{t+1}\xi_{-1} + F^t v_0 + \ldots + Fv_{t-1} + v_t$$

$$\begin{bmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \end{bmatrix} = F^{t+1}\begin{bmatrix} y_{-1} \\ y_{-2} \\ \vdots \\ y_{-p} \end{bmatrix} + F^t \begin{bmatrix} w_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \ldots + F\begin{bmatrix} w_{t-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} w_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$F^{t+1} = F \times F \times \ldots F$$

Extracting the first equation:

$$u_1' \begin{bmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \end{bmatrix} = u_1' F^{t+1} \begin{bmatrix} y_{-1} \\ y_{-2} \\ \vdots \\ y_{-p} \end{bmatrix} + \ldots + u_1' F \begin{bmatrix} w_{t-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + u_1' \begin{bmatrix} w_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$u_1' = [1, 0, \ldots, 0]$$

$$y_t = f_{11}^{(t+1)} y_{-1} + f_{12}^{(t+1)} y_{-2} + \ldots + f_{1p}^{(t+1)} y_{-p} + f_{11}^{(t)} w_0 + \ldots + f_{11} w_{t-1} + w_t$$

where

- $f_{11}^{(t+1)}$ denotes the $(1,1)$ element of F^{t+1},
- $f_{12}^{(t+1)}$ denotes the $(1,2)$ element of F^{t+1},
- etc.

This describes the value of y_t as a linear function of p initial values of y, $(y_{-1}, y_{-2}, \ldots, y_{-p})$ and the history of the input variable w since time 0, $(w_0, w_1, \ldots, w_t)$.
For time $t+j$,

$$\xi_{t+j} = F^{j+1}\xi_{t-1} + F^j v_t + \ldots + F v_{t+j-1} + v_{t+j}$$

$$\begin{aligned} y_{t+j} &= f_{11}^{(j+1)} y_{t-1} + f_{12}^{(j+1)} y_{t-2} + \ldots + f_{1p}^{(j+1)} y_{t-p} \\ &\quad + f_{11}^{(j)} w_t + f_{11}^{(j-1)} w_{t+1} + \ldots + f_{11} w_{t+1} + w_{t+j} \end{aligned}$$

The dynamic multiplier

$$\frac{\partial y_{t+j}}{\partial w_t} = f_{11}^{(j)}$$

when $j = 1$, $f_{11}^{(j)} = \phi_1$. For any pth-order system, the effect on y_{t+1} of a one-unit increase in w_t is given by the coefficient relating y_t to y_{t-1}

$$\begin{aligned} \frac{\partial y_{t+1}}{\partial w_t} &= \phi_1 \\ \frac{\partial y_{t+2}}{\partial w_t} &= \phi_1^2 + \phi_2 = f_{11}^{(2)} \end{aligned}$$

Analytical characterization of $\frac{\partial y_{t+j}}{\partial w_t}$ in terms of eigenvalues of the matrix F. The eigenvalues are

$$|F - \lambda I_p| = 0 \tag{1}$$

The determinant is a p-th order polynomial in λ whose p solutions characterize the p eigenvalues of F.

Proposition

The eigenvalues of F are the values of λ that satisfy

$$\lambda^p - \phi_1 \lambda^{p-1} - \phi_2 \lambda^{p-2} - \ldots - \phi_{p-1} \lambda - \phi_p = 0$$

once we know the eigenvalues, it is straightforward to characterize the dynamic behavior of the system.

Example: $p = 2$, the Eigenvalues are the solutions to

$$\left| \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$-(\phi_1 - \lambda)\lambda - \phi_2 = 0$$

$$\lambda^2 - \phi_1 \lambda - \phi_2 = 0$$

The two eigenvalues of $\mathbf{F}$ for a second-order difference equation are thus given by

$$\lambda_1 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

$$\lambda_2 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

When the eigenvalues of $\mathbf{F}$ are distinct, there exists a nonsingular $(p \times p)$ matrix $\mathbf{T}$ such that

$$\mathbf{F} = \mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1}$$

$\mathbf{\Lambda}$ is a $(p \times p)$ diagonal matrix with the eigenvalues of $\mathbf{F}$

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_p \end{bmatrix}$$

$$\mathbf{F}^j = \mathbf{T}\mathbf{\Lambda}^j\mathbf{T}^{-1}$$

where

$$\mathbf{\Lambda}^j = \begin{bmatrix} \lambda_1^j & 0 & 0 & \dots & 0 \\ 0 & \lambda_2^j & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_p^j \end{bmatrix}$$

$p = 2$

$$\mathbf{F}^j = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} t^{11} & t^{12} \\ t^{21} & t^{22} \end{bmatrix}$$

$$\mathbf{F}^j = \begin{bmatrix} t_{11}\lambda_1^j & t_{12}\lambda_2^j \\ t_{21}\lambda_1^j & t_{22}\lambda_2^j \end{bmatrix} \begin{bmatrix} t^{11} & t^{12} \\ t^{21} & t^{22} \end{bmatrix}$$

$$f_{11}^{(j)} = t_{11}t^{11}\lambda_1^j + t_{12}t^{21}\lambda_2^j$$

$$c_i = t_{1i} t^{i1} \quad i = 1, 2$$

Given that $\mathbf{T}\mathbf{T}^{-1} = \mathbf{I}$

$$c_1 + c_2 = 1$$

The dynamic multiplier can be written as:

$$\frac{\partial y_{t+j}}{\partial w_t} = f_{11}^{(j)} = c_1 \lambda_1^j + c_2 \lambda_2^j$$

It is a weighted average of each of the p eigenvalues raised to the $j-$th power. As j becomes larger the pattern is dominated by the larger eigenvalue.

Distinct eigenvalues

- All of the eigenvalues are less than 1 in absolute value, then the system is stable and its dynamics are represented as a weighted average of decaying exponentials or decaying exponentials oscillating in sign.

- If the eigenvalues are **real** but at least one is greater than unity in absolute value, the system is explosive (e.g.,$|\lambda_1| > 1$) :

$$\lim_{j \to \infty} \frac{\partial y_{t+j}}{\partial w_t} \frac{1}{\lambda_1^j} = c_1$$

- If some of the eigenvalues are **complex** they are conjugate and we have interesting dynamics. For instance,

$$\begin{aligned}\lambda_1 &= a + ib \\ \lambda_2 &= a - ib\end{aligned}$$

where $i = \sqrt{-1}$ and $a, b \in \mathbb{R}$ with modulus $R = \sqrt{\lambda_1 \lambda_2} = \sqrt{a^2 + b^2}$. The modulus R is a real number to be interpreted as the radial distance of z from the origin in the complex plane, in which a and b are measured on the coordinates axes.

Example $p = 2$

$$\begin{aligned} a &= \phi_1/2 \\ b &= \sqrt{-\phi_1^2 - 4\phi_2}/2 \end{aligned}$$

The contribution to the dynamic multiplier $\frac{\partial y_{t+j}}{\partial w_t}$

$$c_1 \lambda_1^j$$

Polar form

$$\lambda_1 = R[\cos(\theta) + i \sin(\theta)] = R[\exp(i\theta)]$$

$$\begin{aligned} R &= \sqrt{a^2 + b^2} \\ \cos(\theta) &= a/R \\ \sin(\theta) &= b/R \end{aligned}$$

$$\lambda_1^j = R^j(\exp(i\theta j)) = R^j[\cos(\theta j) + i\sin(\theta j)]$$
$$\lambda_2^j = R^j(\exp(-i\theta j)) = R^j[\cos(\theta j) - i\sin(\theta j)]$$

The dynamic multiplier

$$\begin{aligned}\frac{\partial y_{t+j}}{\partial w_t} &= f_{11}^{(j)} = c_1\lambda_1^j + c_2\lambda_2^j \\ &= c_1 R^j[\cos(\theta j) + i\sin(\theta j)] + c_2 R^j[\cos(\theta j) - i\sin(\theta j)] \\ &= (c_1 + c_2)R^j\cos(\theta j) + i(c_1 - c_2)R^j\sin(\theta j)\end{aligned}$$

If λ_1, λ_2 are complex conjugates then c_1, c_2 are complex conjugates too.

$$\begin{aligned}c_1 &= \alpha + \beta i \\ c_2 &= \alpha - \beta i\end{aligned}$$

$$
\begin{aligned}
c_1\lambda_1^j + c_2\lambda_2^j &= (c_1+c_2)R^j\cos(\theta j) + i(c_1-c_2)R^j\sin(\theta j) \\
&= [(\alpha+\beta i)+(\alpha-\beta i)]R^j\cos(\theta j) + \\
&\quad i[(\alpha+\beta i)-(\alpha-\beta i)]R^j\sin(\theta j) \\
&= 2\alpha R^j\cos(\theta j) - 2\beta R^j\sin(\theta j)
\end{aligned}
$$

which is strictly real.

If

1. $R = 1$. The multipliers are periodic sine and cosine functions of j. A given increase in w_t increases y_{t+j} for some ranges of j and decreases y_{t+j} over other ranges, with the impulse never dying out as $j \to \infty$.

2. $R < 1$. The impulse follows a sinusoidal pattern though its amplitude decays at the rate R^j.

3. $R > 1$. The amplitude of the sinusoids explodes at the rate R^j.

Complex eigenvalues The eigenvalues are complex when

$$\phi_1^2 + 4\phi_2^2 < 0$$

The modulus

$$\begin{aligned} R^2 &= a^2 + b^2 \\ &= (\phi_1/2)^2 - (\phi_1^2 + 4\phi_2)/4 = -\phi_2 \end{aligned}$$

The system is explosive when

$$R = \sqrt{(-\phi_2)} > 1$$

$$\phi_2 < -1$$

The frequency of oscillations is given by

$$\theta = cos^{-1}(a/R) = cos^{-1}[\phi_1/(2\sqrt{-\phi_2})]$$

Real eigenvalues The larger eigenvalue λ_1 will be > 1 whenever

$$\lambda_1 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} > 1$$

$$\sqrt{\phi_1^2 + 4\phi_2} > 2 - \phi_1$$

Assuming λ_1 real the inequality is satisfied for any

$$\phi_1 > 2$$

If

$$\phi_1 < 2$$

then $\lambda_1 > 1$ when

$$\phi_2 > 1 - \phi_1$$

The smaller eigenvalue, λ_2 will < -1 whenever

$$\lambda_2 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2} < -1$$

$$\sqrt{\phi_1^2 + 4\phi_2} < 2 + \phi_1$$

$$\lambda_2 < -1 \quad \text{if} \quad \begin{cases} \phi_1 < -2 \\ \text{or} \\ \phi_2 > 1 + \phi_1 \end{cases}$$

Suppose that y_t is a stochastic process. Then we define L such that

$$\begin{aligned} Ly_t &= y_{t-1} \\ L^j y_t &= y_{t-j} \ \forall j \in \mathbb{N} \end{aligned}$$

$$L(\beta' x_t) = \beta' L x_t$$

Distributive over the addition operator

$$L(y_t + x_t) = y_{t-1} + x_{t-1}$$

The lag operator follows exactly the same algebraic rules as the multiplicative operator. Polynomial in L

$$(aL + bL^2)$$

The difference operator:

$$\Delta = 1 - L$$

$$\Delta y_t = (1 - L)y_t = y_t - y_{t-1}$$

The n-period difference operator

$$\Delta_n = 1 - L^n.$$

$$\Delta_n y_t = (1 - L^n)y_t = y_t - y_{t-n}.$$

The nth-order difference operator

$$\Delta^n = (1 - L)^n$$

$$\begin{aligned} \Delta^2 y_t &= (1 - L)^2 y_t = \Delta\Delta y_t = (1 - L)\Delta y_t \\ \Delta^n y_t &= (1 - L)^n y_t. \end{aligned}$$

L is an operator and not a variable.

The lag operator allows a great economy of notation in operations on dynamic time series models.

In considering the properties of the polynomials in the lag operator we rather prefer to describe the properties of polynomials in the complex variable z, having the form $z = a + ib$.

The properties derived for polynomials in z can be used directly to interpret the effects of lag operators.

$(1 + \alpha_1 L)$

$$(1 + \alpha_1 z) \;\; \alpha_1 \in \mathbb{R}$$

Postulate the inverse $(1 + \alpha_1 z)^{-1}$ exists: $\delta(x) = (1 + \alpha_1 z)^{-1}$, i.e.

$$\delta(z)(1 + \alpha_1 z) = 1$$

Conjecture $\delta(z)$ is a polynomial of indeterminate order

$$\delta(z) = \delta_0 + \delta_1 z + \delta_2 z^2 + \dots$$

$\delta_i, i = 1, 2, \dots$ are constants to be determined

$$\begin{aligned} \delta(z)(1 + \alpha_1 z) &= (\delta_0 + \delta_1 z + \delta_2 z^2 + \dots)(1 + \alpha_1 z) \\ &= \delta_0 + (\delta_1 + \delta_0 \alpha_1) z + (\delta_2 + \delta_1 \alpha_1) z^2 + \dots \end{aligned}$$

To satisfy the identity $\delta(z)(1 + \alpha_1 z) = 1$ requires $\delta_0 = 1$ and

$$\delta_j = -\delta_{j-1} \alpha_1$$

The existence of the inverse depends on $|\alpha_1|$:

- If $|\alpha_1| < 1$ the terms in $\delta(z)$ form a convergent summable series,

$$\delta(z) = 1 - \alpha_1 z + \alpha_1^2 z^2 - \alpha_1^3 z^3 + \ldots$$

  this series is convergent (i.e. the terms have a finite sum) for any z in the unit circle ($|z| \leq 1$)

- If $|\alpha_1| \geq 1$ the series $1 - \alpha_1 z + \alpha_1^2 z^2 - \alpha_1^3 z^3 + \ldots$ is infinite for at least some such points.

The single root of this polynomial is $-\alpha_1^{-1}$:

$$1 + \alpha_1 z = 0 \rightarrow z = -\frac{1}{\alpha_1}. \tag{2}$$

The condition $|z| > 1$, equivalent to $|\alpha_1| < 1$, is called **invertibility condition** for the polynomial. The inverse function is finite for all $|z| < |\alpha_1|^{-1}$.

$$
\begin{aligned}
y_t &= \phi y_{t-1} + w_t \\
&= \phi L y_t + w_t
\end{aligned}
$$

$$(1 - \phi L) y_t = w_t$$

Multiply both sides by

$$(1 + \phi L + \phi^2 L^2 + \ldots + \phi^t L^t)$$

$$(1+\phi L+\phi^2 L^2+\ldots+\phi^t L^t)(1-\phi L) y_t = (1+\phi L+\phi^2 L^2+\ldots+\phi^t L^t) w_t$$

The compound operator results in

$$
\begin{aligned}
(1 + \phi L + \ldots + \phi^t L^t)(1 - \phi L) &= (1 + \phi L + \ldots + \phi^t L^t) - (1 + \phi L + \ldots + \phi^t L^t \\
&= 1 - \phi^{t+1} L^{t+1}
\end{aligned}
$$

Then

$$(1 - \phi^{t+1} L^{t+1}) y_t = (1 + \phi L + \phi^2 L^2 + \ldots + \phi^t L^t) w_t$$

$$y_t - \phi^{t+1} y_{t-(t+1)} = w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \ldots + \phi^t w_0$$

Exactly the same result of recursive substitution. As t becomes large, $|\phi| < 1$ and y_{-1} finite:

$$y_t - \phi^{t+1} y_{-1} \cong y_t$$

When $|\phi| < 1$ we can think of

$$(1 + \phi L + \phi^2 L^2 + \ldots + \phi^j L^j)$$

as approximating the inverse of $(1 - \phi L)$, with

$$(1 - \phi L)^{-1} = \lim_{j \to \infty} (1 + \phi L + \phi^2 L^2 + \ldots + \phi^j L^j)$$

this operator has the property:

$$(1 - \phi L)^{-1}(1 - \phi L) = 1$$

A sequence is said *bounded* if there exists a finite number $\overline{y}$ such that

$$|y_t| < \overline{y} \quad \forall t.$$

For stochastic sequences: mean square convergence and stationary processes in place of limits of bounded deterministic sequences.

With $|\phi| < 1$ and bounded sequences (stationary processes)

$$y_t = (1 - \phi L)^{-1} w_t$$

$$y_t = w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \dots$$

Aggiungere conclusione

Higher order polynomials. Solution: Factorization. For the second order case, the factorization is:

$$\alpha(z) = 1 + \alpha_1 z + \alpha_2 z^2 = (1 - \mu_1 z)(1 - \mu_2 z)$$

where

$$\begin{aligned} \alpha_1 &= -(\mu_1 + \mu_2) \\ \alpha_2 &= (\mu_1 \mu_2) \end{aligned}$$

The roots are μ_1^{-1}, μ_2^{-1} may be either real or complex conjugate pair. If the roots are outside the unit circle then

$$|\mu_1| < 1, \quad |\mu_2| < 1$$

and

$$\frac{1}{1+\alpha_1 z + \alpha_2 z^2} = \frac{1}{(1-\mu_1 z)} \frac{1}{(1-\mu_2 z)} = \sum_{j=0}^{\infty} \mu_1^j z^j \sum_{j=0}^{\infty} \mu_2^j z^j$$

both of the series in this product are convergent for points z in the unit disk since (applying the triangle inequality)

$$\left| \sum_{j=0}^{\infty} \mu_k^j z^j \right| \leq \sum_{j=0} |\mu_k|^j \, |z|^j < \infty, \quad k = 1, 2$$

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p} + w_t$$

$$(1 - \phi_1 L - \ldots - \phi_p L^p) y_t = w_t$$

Factorizing the operator

$$(1 - \phi_1 L - \ldots - \phi_p L^p) = (1 - \lambda_1 L)(1 - \lambda_2 L) \ldots (1 - \lambda_p L)$$

This is the same as finding the values of $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ such that the following polynomials are the same for all z:

$$(1 - \phi_1 z - \ldots - \phi_p z^p) = (1 - \lambda_1 z)(1 - \lambda_2 z) \ldots (1 - \lambda_p z)$$

Multiply both sides by z^{-p} and define $\lambda = z^{-1}$

$$(z^{-p} - \phi_1 z^{1-p} - \ldots - \phi_p) = (z^{-1} - \lambda_1)(z^{-p} - \lambda_2)\ldots(z^{-1} - \lambda_p)$$

$$(\lambda^p - \phi_1 \lambda^{p-1} - \ldots - \phi_{p-1}\lambda - \phi_p) = (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_p)$$

setting $\lambda = \lambda_i, \quad i = 1, 2, \ldots, p$

$$(\lambda^p - \phi_1 \lambda^{p-1} - \ldots - \phi_{p-1}\lambda - \phi_p) = 0$$

This expression is identical to that which characterizes the eigenvalues of the matrix $\mathbf{F}$.

Factoring a p-th order polynomial in the lag operator,

$$(1 - \phi_1 L - \ldots - \phi_p L^p) = (1 - \lambda_1 L)(1 - \lambda_2 L)\ldots(1 - \lambda_p L)$$

is the same calculation as finding the eigenvalues of the matrix $\mathbf{F}$ of the companion form. The eigenvalues are the same of the parameters in the factorization and are given by the solutions to

$$(\lambda^p - \phi_1 \lambda^{p-1} - \ldots - \phi_{p-1}\lambda - \phi_p) = 0$$

The difference equation is stable if the eigenvalues lie inside the unit circle, or equivalently if the roots of

$$(1 - \phi_1 z - \ldots - \phi_p z^p) = 0$$

lie outside the unit circle.