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Deterministic and Stochastic Trends

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Covariance-stationary time series:

$$Y_t = \mu + \psi(L)\varepsilon_t$$

$$\sum_{i=1}^{\infty} |\psi_i| < \infty \quad \psi_0 = 1$$

$$\begin{aligned} E(\varepsilon_t) &= 0 \\ E\left(\varepsilon_t^2\right) &= \sigma^2 \\ E(\varepsilon_t \varepsilon_s) &= 0 \quad t \neq s \end{aligned}$$

$$E(Y_t) = \mu$$

$$E(Y_{t+s} | Y_t, Y_{t-1}, \ldots) = \widehat{Y}_{t+s|t}$$

This model is unable to capture the features of economic time series, that are characterised by trends.

One reason macroeconomists got interested in unit roots is the question of how to represent trends in time series.
Until the late 70's it was common to simply fit a linear trend to log GNP (by OLS), and then define the stochastic part of the time series as deviations from this trend.

Macroeconomists started to wonder whether shocks to GNP might not more closely resemble the permanent shocks of a random walk more than the transitory shocks of the old AR(2) about a linear trend.

Two possible approaches to time trends in economic time series:

1. **Trend stationarity**

$$Y_t = \alpha + \delta t + \psi(L)\varepsilon_t$$

2. **Unit root process**

$$(1 - L)Y_t = \delta + \psi(L)\varepsilon_t \quad \psi(1) \neq 0$$

With this hypothesis we rule out the case that Y_t is stationary before the differentiation.

If the process is stationary before differentiation, i.e.

$$Y_t = \mu + \chi(L)\varepsilon_t$$

$$\begin{aligned}(1-L)Y_t &= (1-1)\mu + (1-L)\chi(L)\varepsilon_t \\ &= \psi(L)\varepsilon_t\end{aligned}$$

where

$$\psi(L) = (1-L)\chi(L)$$

If the original series is stationary then ΔY_t is stationary too, however

$$\psi(1) = (1-1)\chi(1) = 0$$

so contradicting the hypothesis $\psi(1) \neq 0$.

Unit root process: Random walk with drift, $\psi(L) = 1$

$$Y_t = Y_{t-1} + \delta + \varepsilon_t$$

Encompassing Model (Campbell and Perron (1991)):

$$Y_t = \alpha + \delta t + u_t$$

where $u_t \sim ARMA(p,q)$, $E(u_t) = 0$

$$(1 - \phi_1 L - \cdots - \phi_p L^p) u_t = \left(1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q\right) \varepsilon_t$$

$\varepsilon_t \sim WN$, where $\theta(L)$ is invertible.

Let's suppose that the autoregressive polynomial be factorized as

$$\left(1-\phi_1 L-\cdots-\phi_p L^p\right)=\left(1-\lambda_1 L\right)\left(1-\lambda_2 L\right)\ldots\left(1-\lambda_p L\right)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the eigenvalues

$$u_t=\frac{1+\theta_1 L+\theta_2 L^2+\cdots+\theta_q L^q}{\left(1-\lambda_1 L\right)\left(1-\lambda_2 L\right)\ldots\left(1-\lambda_p L\right)} \varepsilon_t \equiv \psi(L) \varepsilon_t$$

with $\sum_{i=1}^{\infty}\left|\psi_i\right|<\infty$ and roots of $\psi(z)=0$ are outside the unit circle.

So when $\left|\lambda_i\right|<1$ for all i, this becomes a particular case of the process with deterministic trend.

If, instead, $\lambda_1 = 1$ and $|\lambda_i| < 1$ for $i = 2, 3, \ldots, p$ then

$$\phi(L) u_t = \theta(L) \varepsilon_t$$

this implies

$$(1 - L) u_t = \frac{1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q}{(1 - \lambda_2 L) \ldots (1 - \lambda_p L)} \varepsilon_t \equiv \psi^*(L) \varepsilon_t$$

with $\sum_{i=1}^{\infty} |\psi_i^*| < \infty$ and the roots of $\psi^*(L) = 0$ outside the unit circle. In this way if

$$(1 - L) Y_t = (1 - 1)\alpha + (\delta t - \delta t + \delta) + (1 - L) u_t = 0 + \delta + \psi^*(L) \varepsilon_t$$

that takes the form of unit root processes

$$(1 - L) Y_t = \delta + \psi^*(L) \varepsilon_t$$

Unit root because one or more eigenvalues of the autoregressive polynomial lie on the unit circle.

Integrated process because:

$$\begin{aligned} \Delta Y_t &= x_t \\ Y_t &= \sum_t x_t \end{aligned}$$

When we have more than 2 eigenvalues that lie on the unit circle we have to differentiate twice the series in order to achieve stationarity.

Definition (Engle and Granger (1987)). A series with no deterministic component that has a stationary, invertible ARMA representation after differencing d times is said to be *integrated of order d*, which is denoted as $y_t \sim I(d)$.

If $x_t \sim I(0)$ and $z_t \sim I(1)$ then

$$y_t = x_t + z_t \sim I(1)$$

If $y_t \sim I(d)$ then

$$\alpha + \beta y_t \sim I(d)$$

$$y_t = y_{t-s} + \epsilon_t$$

If s equals a seasonal frequency of the series, then y_t is determined by its prior seasonal values plus noise. The filter operator for seasonal unit roots is

$$\begin{aligned}\Delta_s &= (1 - L^s) \\ &= (1 - L)(1 + L + \ldots + L^{s-1}) = \Delta S(L)\end{aligned}$$

$S(L)$ is the moving average seasonal filter.
Definition. Seasonally integrated series (Engle, Granger and Hallman (1988)). A variable $\{y_t\}$ is said to be seasonally integrated of orders d and D, which are denoted as $SI(d, D)$, if $\Delta^d S(L)^D y_t$ is stationary.

If a quarterly series $\Delta_4 y_t$ is stationary, then $\{y_t\}$ is SI(1, 1).

Suppose the series $\{y_t\}$ has been generated by a RW with drift:

$$y_t = \mu + y_{t-1} + \epsilon_t$$

and we the fit an AR(1) to the detrended observations:

$$(1 - \phi L)(y_t - bt) = \epsilon_t$$

which imply that we have:

$$y_t = \phi b + (1 - \phi)bt + \phi y_{t-1} + \epsilon_t$$

or

$$y_t = \alpha + \delta t + \phi y_{t-1} + \epsilon_t$$

An ARMA(p,q)-model could be fitted to the differenced series.

This model class is termed the Autoregressive Integrated Moving Moverage (ARIMA)(p,d,q), where d refers to the order of integration; i.e., how many times the original series must be differenced until a stationary one is obtained.

ARIMA(p,d,q):

$$\phi(L)(1-L)^d y_t = \theta(L)\epsilon_t$$

where $\phi(1) \neq 1$ and $\theta(1) \neq 1$.

The method used for detrending depends on whether the time series is *trend-stationary process* (TSP) (process that is stationary around a trend) or *difference-stationary process* (DSP) (process that is stationary in first differences). One summary conclusion that emerges from the empirical work is that the evidence in favor of deterministic trends is stronger for real variables than for nominal variables.

- If the time series is DSP and we treat it as TSP, underdifferencing.
- If the time series is TSP and we treat it as DSP, overdifferencing.

The serial correlation of the resulting disturbances from the misspecified processes need to be considered. For instance, if the the regression relationship is correctly specified in first differences

$$\Delta Y_t = \beta \Delta X_t + \varepsilon_t$$

this implies

$$Y_t = \alpha + \beta X_t + u_t$$

$$u_t = \varepsilon_t + \varepsilon_{t-1} + \cdots$$

is serially correlated and nonstationary.

On the other hand, if the regression relationship is correctly specified

$$Y_t = \alpha + \beta X_t + v_t$$

this implies that

$$\Delta Y_t = \beta \Delta X_t + v_t - v_{t-1}$$

The errors follow a noninvertible moving-average process.

Beveridge and Nelson (1981) show that any ARIMA(p,1,q) model can be represented as a stochastic trend plus a stationary component. ARIMA(p,1,q) model, Wold representation

$$\Delta y_t = \psi(L)\varepsilon_t$$

$$\Delta y_t = \{\psi(1) + [\psi(L) - \psi(1)]\}\varepsilon_t$$

we can show that

$$\psi(L) - \psi(1) = \psi^*(L)(1 - L)$$

$\psi^* (L)$ has no roots inside or on the unit circle: $|\psi^* (z)| = 0,\ |z| > 1$.

$$
\begin{aligned}
y_t &= y_{t-1} + \{\psi(1) + \psi^*(L)(1-L)\}\,\varepsilon_t \\
y_t &= y_{t-1} + \psi(1)\,\varepsilon_t + \psi^*(L)\,\Delta\varepsilon_t
\end{aligned}
$$

recursive substitution

$$
\begin{aligned}
y_t &= y_{t-2} + \psi(1)\,\varepsilon_{t-1} + \psi^*(L)\,\Delta\varepsilon_{t-1} \\
&\ \vdots \\
y_1 &= y_0 + \psi(1)\,\varepsilon_1 + \psi^*(L)\,\Delta\varepsilon_1
\end{aligned}
$$

$$y_t = y_0 + \psi(1) \sum_{s=1}^{t} \varepsilon_s + \psi^*(L) \sum_{s=1}^{t} \Delta\varepsilon_s$$

$$\sum_{s=1}^{t} \Delta\varepsilon_s = \Delta \sum_{s=1}^{t} \varepsilon_s = \sum_{s=1}^{t} \varepsilon_s - \sum_{s=0}^{t-1} \varepsilon_s = \varepsilon_t$$

assuming $\varepsilon_0 = 0$. The final expression is

$$y_t = y_0 + \psi(1) \sum_{s=1}^{t} \varepsilon_s + \psi^*(L) \varepsilon_t$$

$$E(y_t) = y_0$$

- *Permanent component*: $\psi(1)\sum_{s=1}^{t}\varepsilon_s$. All the past innovations enter in the permanent component.
- *Transitory component*: $\psi^*(L)\varepsilon_t$.

When there is a deterministic trend, with $y_0 = \varepsilon_0 = 0$:

$$\Delta y_t = b + \psi(L)\,\varepsilon_t$$

$$y_t = bt + \psi(1)\sum_{s=1}^{t}\varepsilon_s + \psi^*(L)\,\varepsilon_t$$

- *Deterministic trend*: bt
- *Stochastic Trend*: $\psi(1)\sum_{s=1}^{t}\varepsilon_s$. The stochastic trend incorporates all the random shocks (ε_1 to ε_t) that have permanent effects on the level of y_t.
- *Transitory component*: $\psi^*(L)\,\varepsilon_t$.

$ARIMA\,(0,1,1)$ model with drift

$$\Delta y_t = b + \varepsilon_t + \psi_1 \varepsilon_{t-1}$$

$$\varepsilon_t \sim WN$$

$$y_0 = \varepsilon_0 = 0$$

$$y_t = y_{t-1} + b + \varepsilon_t + \psi_1 \varepsilon_{t-1}$$

$$y_t = (y_{t-2} + b + \varepsilon_{t-1} + \psi_1 \varepsilon_{t-2}) + b + \varepsilon_t + \psi_1 \varepsilon_{t-1}$$

$$y_t = bt + \sum_{s=1}^{t} \varepsilon_s + \psi_1 \sum_{s=1}^{t-1} \varepsilon_s$$

$$y_t = bt + (1 + \psi_1) \sum_{s=1}^{t} \varepsilon_s - \psi_1 \varepsilon_t$$

- Deterministic trend: bt, (DT_t)
- Stochastic Trend: $(1+\psi_1)\sum_{s=1}^{t}\varepsilon_s$, (ST_t)
- Transitory component: $-\psi_1\varepsilon_t$.

The permanent component is the stochastic trend plus the deterministic trend:

$$y_t^p = DT_t + ST_t$$

the permanent component of y_t is a random walk with drift:

$$\begin{aligned} y_t^p &= bt + (1+\psi_1)\sum_{s=1}^{t}\varepsilon_s \\ &= b + b(t-1) + (1+\psi_1)\sum_{s=1}^{t-1}\varepsilon_s + (1+\psi_1)\varepsilon_t \\ &= b + y_t^p + (1+\psi_1)\varepsilon_t. \end{aligned}$$