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No-Arbitrage and Cointegration

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Stochastic trends are prevalent in financial data. Two or more assets might share the same stochastic trend: they are *cointegrated*.

- Exchange rates (Baillie & Bollerslev (1989))
- Foreign currency spot and forward rates (Barnhart and Szakmazy (1991))
- Foreign currency spot and futures rates (Kroner and Sultan (1993))
- Interest rates of different maturities (Engle and Granger (1987))
- etc,..

Forward contract can be priced using a no-arbitrage argument.
At time t:

- Consider a forward contract (FC) that obliges to hand over an amount F at time T to receive an underlying asset.
- The current price is $S(t)$, **spot price**.
- At maturity, we pay F and receive the asset, then worth $S(T)$.
- the profit cannot be known until we know the value $S(T)$.

With a special portfolio of trades we can eliminate all randomness in the future.

- Enter into the forward contract (no costs)
- Simultaneously sell the underlying asset (going short), cash inflow: $+S(t)$
- Net position is zero
- Put the cash in the bank, to receive interest.
- At time T we hand over the amount F and receive the asset.
- Net position at maturity is

$$S(t)e^{r(T-t)} - F$$

Since we start with a portfolio worth zero and we end up with a predictable amount $S(t)e^{r(T-t)} - F$, that predictable amount should also be zero

$$S(t)e^{r(T-t)} - F = 0$$

this entails

$$F = S(t)e^{r(T-t)}$$

to exploit this an make a riskless arbitrage profit:

- Enter into the forward contract (long) (no costs)
- Simultaneously sell the underlying asset (going short): $-S(t)$
- Cash inflow: $+S(t)$
- Net position is zero
- Put the cash in the bank, to receive interest.

- At time T: we hand over the amount F and receive the asset.
- We close the short position
- Net position at maturity is

$$S(t)e^{r(T-t)} - F$$

this is the relationship between the spot price and the forward price. If the relationship is violated then there will be an arbitrage opportunity.

If

$$e^{-r(T-t)}F < S(t)$$

Sell short the asset, buy a forward at t, buy a riskless bond (the bank account).

- At maturity there will be $S(t)e^{r(T-t)}$ in the bank, a short asset and a long forward.
- The asset position cancels when we hand over the amount F
- Profit

$$S(t)e^{r(T-t)} - F$$

If

$$e^{-r(T-t)}F > S(t)$$

We sell short the forward contract, we borrow from the bank $S(t)$ and buy the asset at t

- At maturity there will be a debt of $S(t)e^{r(T-t)}$ with the bank
- At maturity we cash F
- Profit

$$F - S(t)e^{r(T-t)}$$

- No transaction costs
- Two time periods: t_0, t_1

Speculative position in a commodity:

1. Buy a futures contract

$$S_1 - F_{1|0} = \text{Cashflow (futures)}$$

2. Buy the spot commodity and store it

$$S_1 - (1 + R_{1|0})S_0 - W_{1|0}) = \text{Cashflow (storage)}$$

   - $(1 + R_{1|0})S_0$, financing costs
   - $W_{1|0}$, storage cost over the contract period

In a multiperiod economy, the price of a futures contract maturing more than one period ahead may not necessarily equivalent to the stored commodity.

In equilibrium, Cost-of-Carry (C-o-C) hypothesis implies that the return from purchasing a commodity at t and selling it for delivery at $(t+k)$:

$$F_{t+k|t} = (1 + R_{t+k|t})S_t + W_{t+k|t} - C_{t+k|t}$$

1. $F_{t+k|t} - S_t$: *Basis*
2. $S_t R_{t+k|t}$: *Financing or interest costs*
3. $W_{t+k|t}$: *Marginal warehousing costs*
4. $C_{t+k|t}$: *Convenience yield* (liquidity premium, convenience of holding inventories)

Is the forward price un unbiased predictor of the future spot price?

$$F_{t+k|t} = E_t[S_{t+k}]$$

Is the expected risk premium (RP) non-zero?
Two approaches along the lines of the RP hypothesis to explain how risk aversion among hedgers and speculators can affect futures prices and cause them to diverge from the $E_t[S_{t+k}]$.

- Normal backwardation:

  $$F_{t+k|t} < E_t[S_{t+k}]$$

  Keynes: Hedgers are net short in the commodity and the speculators are net long, the futures price will be below the expected future price.

- Normal contango:

$$F_{t+k|t} > E_t[S_{t+k}]$$

- Fama & French (1987):

$$F_{t+k|t} = E_t[S_{t+k}] + E_t[\pi_{t+k|t}]$$

where the $E_t[\pi_{t+k|t}]$ is defined as the bias of $F_{t+k|t}$ over $E_t[S_{t+k}]$. If traders are risk neutral

$$E_t[\pi_{t+k|t}] = 0 \forall t, k$$

Unbiased Expectations Hypothesis.

Two regressions:

$$S_{t+k} - S_t = a_1 + b_1(F_{t+k|t} - S_t) + u_{t+k|t}$$

$$F_{t+k|t} - S_{t+k} = a_2 + b_2(F_{t+k|t} - S_t) + v_{t+k|t}$$

- if $b_1 > 0$ futures price has forecast power for the future spot price
- if $b_2 > 0$ the observed basis at t contains information about the premium to be received at $t + k$, suggesting evidence of time-varying expected RP.

Futures are traded more actively than forward contracts.

- Forward: profit is realized at maturity
- Futures: P&L made on the change in futures price is settled at the end of each trading day by the brokerage house (*marking-to-market*).

Only when the interest rate is non stochastic be equal.

If the interest rate is stochastic and is positively correlated with the spot price of the underlying commodity, the futures price will be greater (less) than the forward price.

Futures contracts are grouped essentially in four categories:

1. physical commodity
2. foreign currency
3. interest rate earning asset
4. stock index

Forward contracts on foreign exchange. *Covered Interest Rate parity.*

Investors will be indifferent between

1. investing in domestic bonds
2. converting domestic funds into foreign-denominated funds at the spot rate, investing in foreign bonds, and converting these funds back into domestic funds at the previously contracted forward rate.

- S_t domestic value of a foreign currency at time t (exchange rate),
- $f_{t|t-k}$ domestic value of a currency forward contract at time $t-k$ that expires at time t,
- $P^d_{t|t-k}$ ($P^f_{t|t-k}$) be the price of a domestic (foreign) pure discount bond at time $t-k$ that pays one dollar at time t.

The forward (futures) and spot prices of an asset are related by

$$\log S_t - \log f_{t|t-1} = c - \log D_{t|t-1} + v_t$$

- S_t spot price at time t
- $f_{t|t-1}$ value of a forward (futures) contract at time $t-1$ which expires at time t
- $D_{t|t-1}$ is the expected net cost-of-carry, or *differential*, over the life of the futures contract

If the differential has a stochastic trend then the spot and futures price do not cointegrate; if the differential is stationary then spot and futures prices are tied together, and they cointegrate.

$$P^d_{t|t-k} = e^{-kr^d_{t|t-k}}, \quad P^f_{t|t-k} = e^{-kr^f_{t|t-k}}$$

where $r^d_{t|t-k}$ ($r^f_{t|t-k}$) is the domestic (foreign) k-period interest rate at time $t-k$.

The no-arbitrage pricing rule is

$$f_{t|t-1} = S_{t-k}\frac{P^f_{t|t-k}}{P^d_{t|t-k}} = S_{t-k}D_{t|t-k}$$

$$D_{t|t-k} = \frac{P^f_{t|t-k}}{P^d_{t|t-k}}$$

is the cost-of-carry or *differential*.

Today's forward price is equal to today's spot price, adjusted by the difference between domestic and foreign interest rates.

Brenner and Kroner JF&QA (1995):

$$\log f_{t|t-k} = \log S_{t-k} + \log D_{t|t-k}$$

- If the $(\log S_{t-k}, \log D_{t|t-k})$ are not cointegrated with cointegrating vector $(1, -1)$ this implies that the forward price has a stochastic trend.
- The forward premium (or basis) $\log S_{t-k} - \log f_{t|t-k}$ is serially correlated if the logarithm of the cost-of-carry is serially correlated. Therefore the persistence of shocks to the forward premium (basis) will be the same as the persistence of shocks to the cost-of-carry.

Arbitrage-based pricing duplicates one asset with a combination of other assets. If the original asset has a stochastic trend, then the duplicated asset should have the same stochastic trend. Hence no-arbitrage pricing can lead to cointegrated asset prices.

If the differential has a stochastic trend, then spot and forward prices will not be cointegrated by themselves: the differential must be included in the system to find cointegration.

If the differential os stationary, the spot price and the forward price can never drift apart.

If the differential has a stochastic trend, then the forward premium (basis) would also have a stochastic trend.

- Many studies examine cointegration between contemporaneous forward and realized spot prices ($\log S_t$ and $\log f_{t|t-k}$).
- While many others examine cointegration between contemporaneous forward and spot prices ($\log S_{t-k}$ and $\log f_{t|t-k}$)

Cointegration exists at any lead and lag of the spot and forward prices.

The time to expiration of the forward contract, k, is fixed, while the time of expiration, t, is changing.

Siqueira & McAleer (1998) specify an ECM representation for the RP hypothesis:

$$\Delta s_{t+1} = b_0 + b_1 \Delta s_t + b_2 \Delta f_{t+1|t} - a(s_t - \beta_1 f_{t|t-1}) + \epsilon_{t+1}$$

- s_{t+1} spot price
- $f_{t+1|t}$ futures price of one-period ahead futures contract at time t
- $r^d_{t+1|t}$ one-period ahead domestic risk free interest rate at time t
- $r^d_{t+1|t}$ one-period ahead foreign risk free interest rate at time t

The orders of integration:

1. All four variables are cointegrated.
2. Two subsets of two I(1) variables are cointegrated.

Four variables are cointegrated:

$$\begin{aligned}\Delta s_{t+1} &= b_0 + b_1 \Delta s_t + b_2 \Delta f_{t+1|t} + b_3' \Delta r^d_{t+1|t} - b_4' \Delta r^f_{t+1|t} \\ &\quad - a'(s_t - \beta_1 f_{t|t-1} - \beta_2' r^d_{t+1|t} - \beta_3' r^f_{t+1|t}) + \epsilon'_{t+1}\end{aligned}$$