Università di Pavia

Il modello partizionato
Il teorema di Frisch-Waugh-Lowell

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Assunzioni: $\mathbf{X}, (N \times K)$, matrice rango-colonna pieno, $r(\mathbf{X}) = K$, $N > K$.

$$\begin{aligned}\mu &= \mathbf{X}\beta \\ &= \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2 \\ &= \mu_1 + \mu_2\end{aligned}$$

$$\begin{array}{cc} \mathbf{X}_1 & (N \times K_1) \\ \mathbf{X}_2 & (N \times K_2) \\ \beta_1 & (K_1 \times 1) \\ \beta_2 & (K_2 \times 1) \end{array}$$

$$K = K_1 + K_2$$

Equazioni normali:

$$(\mathbf{X}'\mathbf{X})\widehat{\beta} = \mathbf{X}'\mathbf{y}$$

$$\begin{bmatrix} \mathbf{X}_1'\mathbf{X}_1 & \mathbf{X}_1'\mathbf{X}_2 \\ \mathbf{X}_2'\mathbf{X}_1 & \mathbf{X}_2'\mathbf{X}_2 \end{bmatrix} \begin{bmatrix} \widehat{\beta}_1 \\ \widehat{\beta}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{X}_1'\mathbf{y} \\ \mathbf{X}_2'\mathbf{y} \end{bmatrix}$$

$$\begin{bmatrix} \widehat{\beta}_1 \\ \widehat{\beta}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{X}_1'\mathbf{X}_1 & \mathbf{X}_1'\mathbf{X}_2 \\ \mathbf{X}_2'\mathbf{X}_1 & \mathbf{X}_2'\mathbf{X}_2 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{X}_1'\mathbf{y} \\ \mathbf{X}_2'\mathbf{y} \end{bmatrix}$$

Usando l'espressione per l'inversa di una matrice a blocchi

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} =$$

$$\begin{bmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{X}_1'\mathbf{X}_1 & \mathbf{X}_1'\mathbf{X}_2 \\ \mathbf{X}_2'\mathbf{X}_1 & \mathbf{X}_2'\mathbf{X}_2 \end{bmatrix}^{-1} = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

$$\begin{aligned} E &= (\mathbf{x}_1'\mathbf{x}_1 - \mathbf{x}_1'\mathbf{x}_2(\mathbf{x}_2'\mathbf{x}_2)^{-1}\mathbf{x}_2'\mathbf{x}_1)^{-1} \\ F &= -(\mathbf{x}_1'\mathbf{x}_1 - \mathbf{x}_1'\mathbf{x}_2(\mathbf{x}_2'\mathbf{x}_2)^{-1}\mathbf{x}_2'\mathbf{x}_1)^{-1}\mathbf{x}_1'\mathbf{x}_2(\mathbf{x}_2'\mathbf{x}_2)^{-1} \\ G &= -(\mathbf{x}_2'\mathbf{x}_2)^{-1}\mathbf{x}_2'\mathbf{x}_1(\mathbf{x}_1'\mathbf{x}_1 - \mathbf{x}_1'\mathbf{x}_2(\mathbf{x}_2'\mathbf{x}_2)^{-1}\mathbf{x}_2'\mathbf{x}_1)^{-1} \\ H &= (\mathbf{x}_2'\mathbf{x}_2)^{-1} + (\mathbf{x}_2'\mathbf{x}_2)^{-1}\mathbf{x}_2'\mathbf{x}_1(\mathbf{x}_1'\mathbf{x}_1 - \mathbf{x}_1'\mathbf{x}_2(\mathbf{x}_2'\mathbf{x}_2)^{-1}\mathbf{x}_2'\mathbf{x}_1)^{-1}\mathbf{x}_1'\mathbf{x}_2(\mathbf{x}_2'\mathbf{x}_2)^{-1} \end{aligned}$$

$$
\begin{aligned}
(\mathbf{X}_1'\mathbf{X}_1 - \mathbf{X}_1'\mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{X}_1)^{-1} &= [\mathbf{X}_1'(\mathbf{I}_N - \mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2')\mathbf{X}_1]^{-1} \\
&= [\mathbf{X}_1'(\mathbf{M}_{X_2})\mathbf{X}_1]^{-1}
\end{aligned}
$$

$$
\begin{aligned}
E &= [\mathbf{X}_1'(\mathbf{M}_{X_2})\mathbf{X}_1]^{-1} \\
F &= -[(\mathbf{X}_1'(\mathbf{M}_{X_2})\mathbf{X}_1)]^{-1}\mathbf{X}_1'\mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-1} \\
G &= -(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{X}_1[\mathbf{X}_1'(\mathbf{M}_{X_2})\mathbf{X}_1]^{-1} \\
H &= (\mathbf{X}_2'\mathbf{X}_2)^{-1} + (\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{X}_1(\mathbf{X}_1'(\mathbf{M}_{X_2})\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-1}
\end{aligned}
$$

$$\widehat{\beta}_1 = E\mathbf{X}_1'\mathbf{y} + F\mathbf{X}_2'\mathbf{y}$$

$$\begin{aligned}
\widehat{\beta}_1 &= [\mathbf{X}_1'(\mathbf{M}_{X_2})\mathbf{X}_1]^{-1}\mathbf{X}_1'\mathbf{y} - [(\mathbf{X}_1'(\mathbf{M}_{X_2})\mathbf{X}_1)]^{-1}\mathbf{X}_1'\mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{y} \\
&= [\mathbf{X}_1'(\mathbf{M}_{X_2})\mathbf{X}_1]^{-1}\mathbf{X}_1'[\mathbf{I} - \mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2']\mathbf{y} \\
&= [\mathbf{X}_1'(\mathbf{M}_{X_2})\mathbf{X}_1]^{-1}\mathbf{X}_1'(\mathbf{M}_{X_2})\mathbf{y}
\end{aligned}$$

Poichè

$$\begin{aligned}
\mathbf{M}_{X_2}' &= \mathbf{M}_{X_2} \\
\mathbf{M}_{X_2}\mathbf{M}_{X_2} &= \mathbf{M}_{X_2}
\end{aligned}$$

$$\widehat{\beta}_1 = [\mathbf{X}_1'\mathbf{M}_{X_2}'\mathbf{M}_{X_2}\mathbf{X}_1]^{-1}\mathbf{X}_1'(\mathbf{M}_{X_2})\mathbf{y}$$

$$\widehat{\beta}_2 = G\mathbf{X}_1'\mathbf{y} + H\mathbf{X}_2'\mathbf{y}$$

$$\begin{aligned}
\widehat{\beta}_2 &= -(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{X}_1[\mathbf{X}_1'(\mathbf{M}_{X_2})\mathbf{X}_1]^{-1}\mathbf{X}_1'\mathbf{y} \\
&\quad +\{(\mathbf{X}_2'\mathbf{X}_2)^{-1} + (\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{X}_1(\mathbf{X}_1'(\mathbf{M}_{X_2})\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-1}\}\mathbf{X}_2'\mathbf{y} \\
&= (\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{y} \\
&\quad -(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{X}_1[\mathbf{X}_1'(\mathbf{M}_{X_2})\mathbf{X}_1]^{-1}\mathbf{X}_1'[\mathbf{I} - \mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2']\mathbf{y} \\
&= (\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{y} - (\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{X}_1[\mathbf{X}_1'(\mathbf{M}_{X_2})\mathbf{X}_1]^{-1}\mathbf{X}_1'(\mathbf{M}_{X_2})\mathbf{y} \\
&= (\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{y} - (\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{X}_1\widehat{\beta}_1 \\
&= (\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'[\mathbf{y} - \mathbf{X}_1\widehat{\beta}_1]
\end{aligned}$$

Frisch e Waugh (1933), Lowell (1963).

$$\mathbf{y} = \mathbf{P}_X \mathbf{y} + \mathbf{M}_X \mathbf{y}$$

$$\mathbf{y} = \mathbf{X}_1 \widehat{\beta}_1 + \mathbf{X}_2 \widehat{\beta}_2 + \mathbf{M}_X \mathbf{y}$$

Premoltiplichiamo con $\mathbf{X}_1' \mathbf{M}_{X_2}$:

$$\mathbf{X}_1' \mathbf{M}_{X_2} \mathbf{y} = \mathbf{X}_1' \mathbf{M}_{X_2} \mathbf{X}_1 \widehat{\beta}_1 + \mathbf{X}_1' \mathbf{M}_{X_2} \mathbf{X}_2 \widehat{\beta}_2 + \mathbf{X}_1' \mathbf{M}_{X_2} \mathbf{M}_X \mathbf{y}$$

ma

$$\mathbf{M}_{X_2} \mathbf{X}_2 = \mathbf{0}$$

$$\mathbf{M}_X \mathbf{M}_{X_2} \mathbf{X}_1 = \mathbf{0}$$

perchè $\mathbf{M}_{X_2} \mathbf{X}_1 \in Col(\mathbf{X})$.

Risolvendo per $\widehat{\beta}_1$ si ottiene

$$\widehat{\beta}_1 = (\mathbf{X}_1' \mathbf{M}_{X_2} \mathbf{X}_1)^{-1} \mathbf{X}_1' \mathbf{M}_{X_2} \mathbf{y}$$

Quindi lo stimatore $\widehat{\beta}_1$ può essere trovato con una procedura a due stadi:

1. regressione di $\mathbf{X}_1$ su $\mathbf{X}_2$, da cui si ottengono i residui $\mathbf{M}_{X_2}\mathbf{X}_1$;
2. regressione di $\mathbf{y}$ sui residui della regressione del primo stadio, $\mathbf{M}_{X_2}\mathbf{X}_1$.

$\widehat{\beta}_1$ cattura la componente di $\mathbf{y}$ collineare con $\mathbf{X}_1$ che non può essere spiegata da $\mathbf{X}_2$.

Se avessimo moltiplicato per $\mathbf{M}_{X_2}$ invece di $\mathbf{X}_1'\mathbf{M}_{X_2}$ avremmo ottenuto:

$$\begin{aligned}\mathbf{M}_{X_2}\mathbf{y} &= \mathbf{M}_{X_2}\mathbf{X}_1\widehat{\beta}_1 + \mathbf{M}_{X_2}\mathbf{X}_2\widehat{\beta}_2 + \mathbf{M}_{X_2}\mathbf{M}_X\mathbf{y} \\ &= \mathbf{M}_{X_2}\mathbf{X}_1\widehat{\beta}_1 + \mathbf{M}_X\mathbf{y}\end{aligned}$$

perchè

$$\mathbf{M}_{X_2}\mathbf{M}_X = (\mathbf{I} - \mathbf{P}_{X_2})\,\mathbf{M}_X = \mathbf{M}_X$$

Questo mostra che la regressione di $\mathbf{M}_{X_2}\mathbf{y}$ su $\mathbf{M}_{X_2}\mathbf{X}_1$ ha gli stessi residui di quella di $\mathbf{y}$ su $\mathbf{X}_1$ e $\mathbf{X}_2$.

Poichè $Col(\mathbf{x}_1)$ e $Col(\mathbf{x}_2)$ non sono in generale sottospazi mutualmente ortogonali, neanche $\mathbf{X}_1\widehat{\beta}_1$ e $\mathbf{X}_2\widehat{\beta}_2$ sono mutualmente ortogonali. Se scomponiamo $\mathbf{X}_1\widehat{\beta}_1$ come segue:

$$\mathbf{X}_1\widehat{\beta}_1 = \mathbf{P}_{X_2}\mathbf{X}_1\widehat{\beta}_1 + \mathbf{M}_{X_2}\mathbf{X}_1\widehat{\beta}_1$$

possiamo esprimere

$$\begin{aligned}\mathbf{y} &= \mathbf{P}_{X_2}\mathbf{X}_1\widehat{\beta}_1 + \mathbf{M}_{X_2}\mathbf{X}_1\widehat{\beta}_1 + \mathbf{X}_2\widehat{\beta}_2 + \mathbf{M}_X\mathbf{y} \\ &= \mathbf{P}_{X_2}\left(\mathbf{X}_1\widehat{\beta}_1 + \mathbf{X}_2\widehat{\beta}_2\right) + \mathbf{M}_{X_2}\mathbf{X}_1\widehat{\beta}_1 + \mathbf{M}_X\mathbf{y} \\ &= \mathbf{P}_{X_2}\left(\mathbf{P}_X\mathbf{y}\right) + \mathbf{M}_{X_2}\mathbf{X}_1\widehat{\beta}_1 + \mathbf{M}_X\mathbf{y}\end{aligned}$$

ma

$$\mathbf{P}_{X_2}(\mathbf{P}_X\mathbf{y}) = \mathbf{P}_{X_2}\mathbf{y}$$

mentre

$$\mathbf{M}_{X_2}\mathbf{X}_1\widehat{\beta}_1 = \mathbf{M}_{X_2}\left(\mathbf{X}_1\widehat{\beta}_1 + \mathbf{X}_2\widehat{\beta}_2\right) = \mathbf{M}_{X_2}\mathbf{P}_X\mathbf{y}$$

quindi

$$\mathbf{y} = \mathbf{P}_{X_2}(\mathbf{P}_X\mathbf{y}) + \mathbf{M}_{X_2}\mathbf{X}_1\widehat{\beta}_1 + \mathbf{M}_X\mathbf{y} = \mathbf{P}_{X_2}\mathbf{y} + \mathbf{M}_{X_2}\mathbf{P}_X\mathbf{y} + \mathbf{M}_X\mathbf{y}$$

$\mathbf{y}$ è la somma di tre componenti mutualmente ortogonali. Infine,

$$\mathbf{M}_{X_2}\mathbf{y} = \mathbf{y} - \mathbf{P}_{X_2}\mathbf{y} = \mathbf{M}_{X_2}\mathbf{P}_X\mathbf{y} + \mathbf{M}_X\mathbf{y}$$

questa scomposizione ortogonale è la scomposizione che si ottiene con la regressione di $\mathbf{M}_{X_2}\mathbf{y}$ sulle colonne di $\mathbf{M}_{X_2}\mathbf{X}_1$.

Infatti:

$$\mathbf{M}_X \mathbf{M}_{X_2} \mathbf{X}_1 = \mathbf{M}_X \mathbf{M}_{X_2} = \mathbf{0}$$

che implica

$$\mathbf{y}' \mathbf{M}_X \mathbf{M}_{X_2} \mathbf{X}_1 = \mathbf{0}.$$

In conclusione,

$$\mathbf{P}_{M_{X_2} X_1} \mathbf{M}_{X_2} \mathbf{y} = \mathbf{M}_{X_2} \mathbf{X}_1 \widehat{\beta}_1$$

$$\mathbf{M}_{M_{X_2} X_1} \mathbf{M}_{X_2} \mathbf{y} = \mathbf{M}_X \mathbf{y}$$

Dato:

$$\mu = \mathbf{X}\beta = \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2$$

Se

$$\mathbf{X}_1'\mathbf{X}_2 = \mathbf{0}_{K_1 \times K_2}$$

allora

$$\begin{bmatrix} \widehat{\beta}_1 \\ \widehat{\beta}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{X}_1'\mathbf{X}_1 & \mathbf{X}_1'\mathbf{X}_2 \\ \mathbf{X}_2'\mathbf{X}_1 & \mathbf{X}_2'\mathbf{X}_2 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{X}_1'\mathbf{y} \\ \mathbf{X}_2'\mathbf{y} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{X}_1'\mathbf{X}_1 & \mathbf{0} \\ \mathbf{0}' & \mathbf{X}_2'\mathbf{X}_2 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{X}_1'\mathbf{y} \\ \mathbf{X}_2'\mathbf{y} \end{bmatrix}$$

$$= \begin{bmatrix} (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{y} \\ (\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{y} \end{bmatrix}$$

Possiamo rendere ortogonali i regressori con la matrice:

$$\mathbf{A} = \begin{bmatrix} \mathbf{I}_{K_1} & \mathbf{0} \\ -(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{X}_1 & \mathbf{I}_{K_2} \end{bmatrix} \qquad (K \times K)$$

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{I}_{K_1} & \mathbf{0} \\ (\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{X}_1 & \mathbf{I}_{K_2} \end{bmatrix}$$

$$\mathbf{XA} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_{K_1} & \mathbf{0} \\ -(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{X}_1 & \mathbf{I}_{K_2} \end{bmatrix}$$

$$\mathbf{XA} = \begin{bmatrix} \mathbf{X}_1 - \mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{X}_1 \vdots & \mathbf{X}_2 \end{bmatrix}$$

Riscriviamo il modello con

$$
\begin{aligned}
\mu &= \mathbf{X}\beta = \mathbf{X}(\mathbf{A}\mathbf{A}^{-1})\beta \\
&= (\mathbf{X}\mathbf{A})(\mathbf{A}^{-1}\beta) \\
&= \mathbf{Z}\delta \\
&= \left[\begin{array}{cc} \mathbf{X}_1 - \mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{X}_1 \vdots & \mathbf{X}_2 \end{array}\right] \left[\begin{array}{c} \delta_1 \\ \delta_2 \end{array}\right] \\
&= \left[\begin{array}{cc} \left(\mathbf{I} - \mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\right)\mathbf{X}_1 \vdots & \mathbf{X}_2 \end{array}\right] \left[\begin{array}{c} \delta_1 \\ \delta_2 \end{array}\right] \\
&= \left[\begin{array}{cc} \mathbf{M}_{X_2}\mathbf{X}_1 \vdots & \mathbf{X}_2 \end{array}\right] \left[\begin{array}{c} \delta_1 \\ \delta_2 \end{array}\right] \\
&= \mathbf{M}_{X_2}\mathbf{X}_1\delta_1 + \mathbf{X}_2\delta_2
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{Z}'\mathbf{Z} &= (\mathbf{XA})'(\mathbf{XA}) \\
&= \mathbf{A}'\mathbf{X}'\mathbf{XA} \\
&= \begin{bmatrix} \mathbf{I}_{K_1} & -\mathbf{X}_1'\mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-1} \\ \mathbf{0}' & \mathbf{I}_{K_2} \end{bmatrix} \begin{bmatrix} \mathbf{X}_1'\mathbf{X}_1 & \mathbf{X}_1'\mathbf{X}_2 \\ \mathbf{X}_2'\mathbf{X}_1 & \mathbf{X}_2'\mathbf{X}_2 \end{bmatrix} \\
& \begin{bmatrix} \mathbf{I}_{K_1} & \mathbf{0} \\ -(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{X}_1 & \mathbf{I}_{K_2} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{X}_1'\mathbf{X}_1 - \mathbf{X}_1'\mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{X}_1 & \mathbf{X}_1'\mathbf{X}_2 - \mathbf{X}_1'\mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{X}_2 \\ \mathbf{X}_2'\mathbf{X}_1 & \mathbf{X}_2'\mathbf{X}_2 \end{bmatrix} \\
& \begin{bmatrix} \mathbf{I}_{K_1} & \mathbf{0}_{K_1 \times K_2} \\ -(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{X}_1 & \mathbf{I}_{K_2} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{Z}'\mathbf{Z} &= \begin{bmatrix} \mathbf{X}_1'(\mathbf{I}_N - \mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2')\mathbf{X}_1 & \mathbf{X}_1'(\mathbf{I}_N - \mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2')\mathbf{X}_2 \\ \mathbf{X}_2'(\mathbf{I}_N - \mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2')\mathbf{X}_1 & \mathbf{X}_2'\mathbf{X}_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{X}_1'(\mathbf{I}_N - \mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2')\mathbf{X}_1 & \mathbf{0}_{K_1 \times K_2} \\ \mathbf{0}'_{K_2 \times K_1} & \mathbf{X}_2'\mathbf{X}_2 \end{bmatrix}
\end{aligned}
$$

La relazione tra δ e β è data da:

$$\delta = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \mathbf{A}^{-1}\beta = \begin{bmatrix} \mathbf{I}_{K_1} & \mathbf{0} \\ (\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{X}_1 & \mathbf{I}_{K_2} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

$$\begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} \beta_1 \\ (\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{X}_1\beta_1 + \beta_2 \end{bmatrix}$$

Infatti

$$\begin{aligned}(\mathbf{X}_1 - \mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{X}_1)\delta_1 + \mathbf{X}_2\delta_2 &= \mathbf{X}_1\beta_1 - \mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{X}_1\beta_1 + \\ &\quad \mathbf{X}_2\left((\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{X}_1\beta_1 + \beta_2\right) \\ &= \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2\end{aligned}$$

$$(\mathbf{Z}'\mathbf{Z})^{-1} = \begin{bmatrix} (\mathbf{X}_1'\mathbf{M}_{X_2}\mathbf{X}_1)^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{X}_2'\mathbf{X}_2)^{-1} \end{bmatrix}$$

$$\widehat{\delta} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{y} = \begin{bmatrix} (\mathbf{X}_1'\mathbf{M}_{X_2}\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{M}_{X_2}\mathbf{y} \\ (\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{y} \end{bmatrix}$$