Università di Pavia

Econometria

2008-2009

Esercizi 2
Soluzioni

Maggio, 2009

1. (MRL semplice) Trovate lo stimatore OLS di β_1 e β_2 nel seguente modello

$$\mu_t = \beta_1 + \beta_2 x_{t2} \quad t = 1, \ldots, N$$

dove

$$x_{t2} = \begin{cases} 1 & t = 1, \ldots, \frac{N}{2} \\ 0 & t = \frac{N}{2} + 1, \ldots, N \end{cases}$$

Soluzione
Usando il teorema di FWL

$$\widehat{\beta}_1 = (\mathbf{X}_1'\mathbf{M}_{X_2}\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{M}_{X_2}\mathbf{y}$$

$$\mathbf{M}_{X_2} = \mathbf{I}_N - \frac{1}{N/2}\mathbf{X}_2\mathbf{X}_2'$$

$$\mathbf{M}_{X_2}\mathbf{X}_1 = \mathbf{X}_1 - \mathbf{X}_2$$

$$\mathbf{X}_1'\mathbf{M}_{X_2}\mathbf{X}_1 = \mathbf{X}_1'\mathbf{X}_1 - \mathbf{X}_1'\mathbf{X}_2 = N - \frac{N}{2} = \frac{N}{2}$$

$$\widehat{\beta}_1 = \frac{2}{N}(\mathbf{X}_1 - \mathbf{X}_2)'\mathbf{y} = \frac{2}{N}\left(\sum_{t=1}^{N} y_t - \sum_{t=1}^{N/2} y_t\right) = \frac{2}{N}\sum_{t=N/2+1}^{N} y_t$$

$$\mathbf{M}_{X_1} = \mathbf{I}_N - \boldsymbol{\iota}(\boldsymbol{\iota}'\boldsymbol{\iota})^{-1}\boldsymbol{\iota}'$$

$$\mathbf{M}_{X_1}\mathbf{X}_2 = (\mathbf{I}_N - \boldsymbol{\iota}(\boldsymbol{\iota}'\boldsymbol{\iota})^{-1}\boldsymbol{\iota}')\mathbf{X}_2 = \mathbf{X}_2 - \frac{1}{2}\boldsymbol{\iota}$$

$$\mathbf{X}_2'\mathbf{M}_{X_1}\mathbf{X}_2 = \mathbf{X}_2'\mathbf{X}_2 - \frac{1}{2}\mathbf{X}_2'\boldsymbol{\iota} = \frac{N}{2} - \frac{N}{4} = \frac{N}{4}$$

$$\widehat{\beta}_2 = (\mathbf{X}_2'\mathbf{M}_{X_1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{M}_{X_1}\mathbf{y} = \frac{4}{N}(\mathbf{X}_2 - \frac{1}{2}\boldsymbol{\iota})'\mathbf{y} = \frac{1}{N/2}\left(\sum_{t=1}^{N/2} y_t - \sum_{t=N/2+1}^{N} y_t\right)$$

2. Considerate la forma quadratica $\mathbf{y}'(\mathbf{I} - \mathbf{P_X})\mathbf{y}$ dove $\mathbf{X}$ è una matrice $(N \times K)$ di rango colonna pieno y è un vettore $(N \times 1)$, e $\mathbf{P_X} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$.

(a) Mostrate che per tutti i $\boldsymbol{\mu} \in \text{Col}(\mathbf{X})$,

$$(\mathbf{y} - \boldsymbol{\mu})'(\mathbf{I} - \mathbf{P_X})(\mathbf{y} - \boldsymbol{\mu}) = \mathbf{y}'(\mathbf{I} - \mathbf{P_X})\mathbf{y}$$

(b) Mostrate che $\mathbf{y}'(\mathbf{I} - \mathbf{P_X})\mathbf{y} \geq 0$

Soluzione

(a) Poichè $\boldsymbol{\mu} \in Col(\mathbf{X})$, quindi $\mathbf{P_X}\boldsymbol{\mu} = \boldsymbol{\mu}$ and $(\mathbf{I}_N - \mathbf{P_X})\boldsymbol{\mu} = \mathbf{0}$. perchè $(\mathbf{I} - \mathbf{P_X})' = \mathbf{I} - \mathbf{P_X}$ e $(\mathbf{I} - \mathbf{P_X})^2 = \mathbf{I} - \mathbf{P_X}$,

$$\begin{aligned}
(\mathbf{y} - \boldsymbol{\mu})'(\mathbf{I}_N - \mathbf{P_X})(\mathbf{y} - \boldsymbol{\mu}) &= (\mathbf{y} - \boldsymbol{\mu})'(\mathbf{I}_N - \mathbf{P_X})^2(\mathbf{y} - \boldsymbol{\mu}) \\
&= (\mathbf{y} - \boldsymbol{\mu})'(\mathbf{I}_N - \mathbf{P_X})'(\mathbf{I}_N - \mathbf{P_X})(\mathbf{y} - \boldsymbol{\mu}) \\
&= \mathbf{y}'(\mathbf{I}_N - \mathbf{P_X})\mathbf{y}
\end{aligned}$$

(b) Mostrate che $\mathbf{y}'(\mathbf{I} - \mathbf{P_X})\mathbf{y} \geq 0$ Usando il risultato al punto precedente:

$$\mathbf{y}'(\mathbf{I}_N - \mathbf{P_X})\mathbf{y} = \mathbf{y}'(\mathbf{I}_N - \mathbf{P_X})'(\mathbf{I}_N - \mathbf{P_X})\mathbf{y} = \|(\mathbf{I}_N - \mathbf{P_X})\mathbf{y}\|^2 \geq 0.$$

3. Dato un campione casuale $\{X_1, X_2, \ldots, X_N\}$ da una popolazione finita con media μ e varianza σ^2 trovate uno stimatore non distorto della varianza e mostratene la correttezza.

Soluzione

Stimatore corretto della varianza quando la media è non nota:

$$\begin{aligned}
s^2 &= \frac{1}{N-1}\sum_{t=1}^{N}(X_t - \overline{X})^2 \\
&= \frac{N}{N-1}\left[\frac{1}{N}\sum_{t=1}^{N}(X_t - \overline{X})^2\right] = \frac{N}{N-1}\widehat{\sigma}^2
\end{aligned}$$

dove $\overline{X} = N^{-1}\sum_t X_t$.

$$\begin{aligned}
E[s^2] &= \frac{N}{N-1}E\left[\frac{1}{N}\sum_{t=1}^{N}(X_t - \overline{X})^2\right] \\
&= \frac{N}{N-1}E\left[\frac{1}{N}\sum_{t=1}^{N}(X_t^2) - (\overline{X})^2\right] \\
&= \frac{N}{N-1}\left[\frac{1}{N}\sum_{t=1}^{N}E(X_t^2) - E(\overline{X}^2)\right]
\end{aligned}$$

con $E(X_t^2) = \sigma^2 + \mu^2$.

$$E[s^2] = \frac{N}{N-1}\left\{\frac{1}{N}N(\sigma^2 + \mu^2) - E\left[\left(\frac{1}{N}\sum_{t=1}^{N}X_t\right)^2\right]\right\}$$

$$
\begin{aligned}
E[s^2] &= \frac{N}{N-1}\left\{(\sigma^2+\mu^2) - \frac{1}{N^2}E\left[\left(\sum_{t=1}^{N} X_t\right)^2\right]\right\} \\
&= \frac{N}{N-1}\left\{(\sigma^2+\mu^2) - \frac{1}{N^2}E\left(\sum_t \sum_s X_t X_s\right)\right\} \\
&= \frac{N}{N-1}\left\{(\sigma^2+\mu^2) - \frac{1}{N^2}\left(N(\sigma^2+\mu^2) + (N^2-N)\mu^2\right)\right\} \\
&= \frac{N}{N-1}\left\{(\sigma^2+\mu^2) - \frac{1}{N^2}\left(N\sigma^2 + N^2\mu^2\right)\right\} \\
&= \frac{N}{N-1}\left(\sigma^2 - \frac{1}{N}\sigma^2\right) = \sigma^2.
\end{aligned}
$$