Stylized Facts for Financial Returns

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Introduction

Discussion of the properties of

- means
- variances
- distributions
- autocorrelations

of returns by referring to empirical evidence.
Three stylized facts:

- The distribution of returns is not normal
- No correlation between returns of different days
- Correlations between the magnitudes of returns on nearby days are positive and statistically significant

These properties can be explained by changes through time in volatility. These stylized facts are pervasive across time as well as across markets.
Population and sample moments

Population moments

- **Skewness:**
  \[
  sk = \frac{E[(r_t - E(r_t))^3]}{\sigma^3}
  \]

- **Kurtosis:**
  \[
  k = \frac{E[(r_t - E(r_t))^4]}{\sigma^4}
  \]

Sample statistics

- **Mean**
  \[
  \bar{r} = \frac{1}{T} \sum_{t=1}^{T} r_t
  \]

- **Standard deviation:**
  \[
  s^2 = \frac{1}{T-1} \sum_{t=1}^{T} (r_t - \bar{r})^2
  \]
Population and sample moments

- Sample Skewness:

\[ sk = \frac{\sum_{t=1}^{T} (r_t - \bar{r})^3}{s^3} / T - 1 \]

- Sample Kurtosis:

\[ k = \frac{\sum_{t=1}^{T} (r_t - \bar{r})^4}{s^4} / T - 1 \]
Test statistic for the null hypothesis that the expected return is zero

\[ z = \frac{\bar{r}}{s/\sqrt{T}} \]

- Skewness statistics are sometimes used to assess the symmetry if distributions
- Kurtosis statistics are often interpreted as a measure of similarity to a normal distribution.
- These statistics are sensitive to extreme observations because they make use of the third and fourth powers of the observations, respectively.
- The standard error of \(sk\) is \(\sqrt{6/T}\) for a random sample from a normal distribution. Evidence of unconditional skewness in US stock returns.
- The standard error of the kurtosis estimate is \(\sqrt{24/T}\) for a random sample from a normal distribution.
1. **The distribution of returns is not normal**
   - It is approximately symmetric
   - It has fat tails (extreme returns)
   - It has a high peak

- There are more observations in the range 
  \[(\bar{r} - \frac{1}{2}s; \bar{r} + \frac{1}{2}s)\]
  than are expected from a normal distribution, corresponding to a high peak in empirical distributions

- More extreme observations, either below \(\bar{r} - 3s\) or above \(\bar{r} + 3s\) \(\rightarrow\) **Fat tails.**
Returns distribution

From the relative frequency of standardized daily NYSE returns, \( z_t = \frac{r_t - \bar{r}}{s} \):

- The average frequency of observations more than 4 standard deviations from the mean is close to 0.4%, this means that \( \text{Pr}\{z_t > 4\} = 0.004 \), i.e. once a year, \( 0.004 \times 252 = 1.008 \).

- Such events would only occur once in sixty years if daily returns were observations from a normal distribution, \( \Phi(|z_t| > 4) = 2 \times 0.000031671 \).

- A standard deviation for a daily stock index return of 1% means that we might expect an index to rise or fall by more than 4% from one market close to the next, very approximately once a year.

- If we use the normal distribution to model the financial returns, we will underestimate the number and magnitude of crashes.
Returns distribution

Many returns are zero and this helps to explain the peaked centers of the empirical distributions.

<table>
<thead>
<tr>
<th>Frequency</th>
<th>( r_t = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coca Cola</td>
<td>6.64</td>
</tr>
<tr>
<td>GE</td>
<td>6.37</td>
</tr>
<tr>
<td>GM</td>
<td>6.56</td>
</tr>
</tbody>
</table>

The US stock zeros are a consequence of the minimum price movement being relatively large relatively compared with the standard deviation of returns.

The average price is about $60 and the minimum price change was one-eighth of a dollar; thus the smallest positive return was typically 0.2% \((60.125/60 - 1 \approx 0.002)\), which is more than 0.1 s.d.

The chance of a normal variable being within 0.05 s.d. of the mean value is 4%, so the number of US stock zeros can be explained by discrete prices combined with a peaked distribution.
There is almost no correlation between returns for different days

- Sample autocorrelation at lag $\tau$:

$$
\hat{\rho}_\tau = \sum_{t=1}^{T-\tau} \frac{\sum_{t=\tau+1}^{T} (r_t - \bar{r})(r_{t-\tau} - \bar{r})}{\sum_{t=1}^{T} (r_t - \bar{r})^2}, \quad \tau > 0
$$

- Large firms have positive first lag autocorrelations
- Portfolio and spot indices have higher estimates than individual securities
- Future on indices have less dependence than spot indices.
- Tests: The autocorrelation estimates can be used to test the null hypothesis that the process generating the observed returns is a series of independent and identically distributed (i.i.d.) random variables.

(Portmanteau $Q$-statistic of Box and Pierce (1970):

$$
Q_k = T \sum_{\tau=1}^{K} \hat{\rho}_\tau^2 \to \chi_k^2
$$
3 Autocorrelations of transformed returns: There is positive dependence between absolute returns on nearby days, and likewise for squared returns.

- $|r_t|^\lambda$, $\lambda = 1, 2$.
- since $\frac{|r_t|^{\lambda-1}}{\lambda} \to \log |r_t|$ as $\lambda \to 0$. But not when $r_t = 0$
- Solution: Logarithmic absolute returns: $l_t = \log (|r_t - \bar{r}|)$. The ACF of $l_t$ will interpreted as the appropriate numbers $\lambda = 0$.
- The autocorrelations of absolute returns are always positive at a lag of 1 day.
- Positive dependence continues to be found for several further lags.
- Power transformations of $|r_t|$, including $r_t^2$, also display positive dependence but generally to a lesser degree.
- Most and often all of these conclusions apply to any long series of daily returns from a financial asset that is traded frequently.
- The high dependence in series of $|r_t|$ proves that the returns process is not made up of i.i.d. random variables.
- Large absolute returns are more likely than small absolute returns to be followed by large absolute returns.
- Changes in price volatility create clusters of high and low volatility.
- During periods of high volatility expected absolute returns are relatively high.
High-frequency data

1. Prices recorded several times each hour generate large datasets

   HF data:
   1. Allows us to learn more about how prices react to information
   2. More observations allow us to estimate and forecast volatility more accurately.
   3. Microstructure effects (spread between buying and selling prices) become more important.
   4. Intraday patterns in trading behavior have to be modeled.
   5. The size of datasets can become daunting.
   6. TAQ (Trades and Quotes) database from the NYSE.
Microstructure: Trading organization at financial markets determine the form of available data.

Price may be determined by either a *quote-driven* or an *order-driven* structure.

These structures differ in the method that establishes a price between buyers and sellers.

*Quotes* are issued by market makers who are prepared to be both buyers and sellers of the asset.

*Orders* can be matched automatically by electronic systems, removing intermediaries from the dealing process.

Recent years: more electronic trading and more use of *order-driven* structures.
High-frequency data

- Prices data are often only available as either *quotes* or *transaction* prices.
- Market maker: *bid* (it buys at the bid price, \( b \)) and *ask* (it sells at the ask price, \( a \)) price.
- The spread \( a - b \) is the profit of the market maker in return for supplying liquidity.
- Order-driven markets: *Order book* contains a set of limit orders from which the most competitive bid and ask prices are determined.
- Transaction prices dataset can reflect a spread even when the market is order-driven.
- Prices may exhibit a *bid-ask bounce* effect for a period of time, during which the best bid and ask are constant with transactions prices bouncing between the two levels: Trades are either at the bid or at the ask price.
- HF datasets include prices and the times at which they are recorded, often accurate to the nearest second. They may also include quantities traded.