CORIPE Piemonte
Master in Economics

Systems of Simultaneous Equations
Identification

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Structural Form – 1

\[ y_t' \Gamma + x_t' B = \varepsilon_t' \]

\[ \Gamma : (g \times g) \]
\[ B : (k \times g) \]

\( y_t \), endogenous variables \((g \times 1)\), \( x_t \) \((k \times 1)\) contains predetermined variables (exogenous, lagged exogenous, lagged endogenous).

Complete system when the number of equations is equal to the number of endogenous variables:

\[ Y \Gamma + X B = \varepsilon \]
Structural Form – 2

Keynesian Model

\[ C_t = \beta_1 + \beta_2 Y_t + \epsilon_{1t} \]
\[ Y_t = C_t + I_t \]
\[ I_t = \gamma_1 + \gamma_2 i_t + \epsilon_{3t} \]

<table>
<thead>
<tr>
<th>Eq</th>
<th>( C_t )</th>
<th>( Y_t )</th>
<th>( I_t )</th>
<th>cons</th>
<th>( i_t )</th>
<th>( \epsilon )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>( C_t )</td>
<td>(-\beta_2 Y_t)</td>
<td>0</td>
<td>(-\beta_1)</td>
<td>0</td>
<td>( \epsilon_{1t} )</td>
</tr>
<tr>
<td>2nd</td>
<td>(-C_t)</td>
<td>( Y_t )</td>
<td>(-I_t)</td>
<td>0</td>
<td>0</td>
<td>( = 0 )</td>
</tr>
<tr>
<td>3rd</td>
<td>0</td>
<td>0</td>
<td>( I_t )</td>
<td>(-\gamma_1)</td>
<td>(-\gamma_2 i_t)</td>
<td>( \epsilon_{3t} )</td>
</tr>
</tbody>
</table>
Structural Form – 3

\[
\begin{bmatrix}
C_t & Y_t & I_t
\end{bmatrix}
\begin{bmatrix}
1 & -\beta_2 & 0 \\
-1 & 1 & -1 \\
0 & 0 & 1
\end{bmatrix}
+ \begin{bmatrix}
1 & i_t
\end{bmatrix}
\begin{bmatrix}
-\beta_1 & 0 & -\gamma_1 \\
0 & 0 & -\gamma_2
\end{bmatrix}
= \begin{bmatrix}
\epsilon_{1t} & 0 & \epsilon_{3t}
\end{bmatrix}
\]

\[y_t' = \begin{bmatrix}
C_t & Y_t & I_t
\end{bmatrix}\]

\[
\Gamma = \begin{bmatrix}
1 & -\beta_2 & 0 \\
-1 & 1 & -1 \\
0 & 0 & 1
\end{bmatrix}
\]

\[x_t' = \begin{bmatrix}
1 & i_t
\end{bmatrix}\]

\[
B = \begin{bmatrix}
-\beta_1 & 0 & -\gamma_1 \\
0 & 0 & -\gamma_2
\end{bmatrix}
\]
\[ \varepsilon'_t = \begin{bmatrix} \varepsilon_{1t} & 0 & \varepsilon_{3t} \end{bmatrix} \]
Structural Form – 4
Assumptions

• $\Gamma$ must be invertible

• $\Gamma$ and $B$ identifiable

• $\text{rank}(X) = k$

• No instantaneous endogenous in $X$

• $E(\varepsilon_t) = 0$
\[ E(\varepsilon_t \varepsilon'_t) = \Sigma \text{ independent of time} \]

\[ E(\varepsilon_t \varepsilon'_{t-k}) = 0 \ \forall k \neq 0. \]
Total number of parameters

\[
\Gamma : \quad g^2 - g \\
\mathcal{B} : \quad k \cdot g \\
\Sigma : \quad \frac{g(g + 1)}{2} \\
\text{Total} : \quad (g^2 - g) + kg + \frac{g(g + 1)}{2}
\]
Reduced Form – 1

\[ y_t' \Gamma + x_t' B = \varepsilon_t' \]

\[ y_t' \Gamma = x_t' (-B) + \varepsilon_t' \]

\[ y_t' = x_t' (-B \Gamma^{-1}) + \varepsilon_t' \Gamma^{-1} \]

\[ y_t' = x_t' \Pi + v_t' \]
Reduced Form – 2

\[ \Pi = -BV\Gamma^{-1} \]
\[ v'_t = \varepsilon'_t \Gamma^{-1} \]
\[ E \left( v_t v'_t \right) = \Omega \]

\[ E \left( v_t v'_t \right) = E \left( \Gamma^{-1'} \varepsilon_t \varepsilon'_t \Gamma^{-1} \right) \]
\[ = \Gamma^{-1'} E \left( \varepsilon_t \varepsilon'_t \right) \Gamma^{-1} \]
\[ = \left( \Gamma' \right)^{-1} E \left( \varepsilon_t \varepsilon'_t \right) \Gamma^{-1} \]

\[ \Omega = \left( \Gamma' \right)^{-1} \Sigma \Gamma^{-1} \]
\[ \Sigma = \Gamma' \Omega \Gamma \]
Reduced Form – 3

\[ \gamma Y + XB = \varepsilon \]

\[
\begin{align*}
Y &= X(-\Gamma^{-1}) + \varepsilon \\
Y &= X\Pi + V
\end{align*}
\]

\[
\begin{align*}
\Pi &: (k \times g) \\
V &: (T \times g)
\end{align*}
\]

The \( i \)-th reduced form equation is:

\[
\begin{align*}
Y_{ui}^c &= X\Pi u_i^c + V u_i^c \\
y^i &= X\pi^i + v^i
\end{align*}
\]

\[ \Pi = [\pi^1, \pi^2, ..., \pi^g] \]
Identification – 1
Assumption: Knowledge of the true $\Pi$ and $\Omega$. Total number of parameters:

$$kg + \frac{g(g + 1)}{2}.$$ 

Structural Form, Total number of parameters: $(g^2 - g) + kg + \frac{g(g+1)}{2}$.

Excess of unknowns with respect to the data: $g^2 - g$. 
Identification – 2
For each reduced form we have an infinite number of structural forms (observationally equivalent)

\[ y_t' \Gamma + x_t' B = \varepsilon_t' \]
\[ y_t' \Gamma R + x_t' BR = \varepsilon_t'R \]
\[ y_t' \Gamma^* + x_t' B^* = \varepsilon_t'^* \]
Identification – 3

The associated reduced form is

\[ y_t' = x_t' \left( -B^* \Gamma^*^{-1} \right) + \varepsilon_t^* \Gamma^*^{-1} \]

\[ y_t' = x_t' \Pi^* + V^* \]

but

\[ \Pi^* = -B^* \Gamma^*^{-1} = -BRR^{-1} \Gamma^{-1} \]

\[ = -B \Gamma^{-1} = \Pi \]

Economic theory must fix some elements of these structural matrices in advance. When there are not sufficient prior restrictions to rule out observationally equivalent structures, the model is said to be underidentified, in whole or in part.
Identification – 4
Consider the first equation of the set:

\[ y_t' \Gamma (u_1^c) + x_t' B (u_1^c) = \varepsilon_t' (u_1^c) \]

- \( g_1 \) : included endogenous variables
- \( g - g_1 \) : excluded endogenous variables
- \( k_1 \) : included predetermined variables
- \( k - k_1 \) : excluded predetermined variables

\[ y_t' = \begin{bmatrix} y_{1t}, \ldots, y_{g1t}, & y_{g1+1t}, \ldots, y_{gt} \end{bmatrix} \]

\begin{bmatrix} \text{included endogenous} & \text{excluded endogenous} \end{bmatrix}
Identification – 5

\[
\Gamma (u^c_1) = \begin{bmatrix}
\gamma_{11} \\
\gamma_{21} \\
\vdots \\
\gamma_{g_1,1} \\
0 \\
\vdots \\
0
\end{bmatrix} = \begin{bmatrix}
\gamma^c_1 \\
0 \\
\end{bmatrix} (g_1 \times 1) \quad ((g - g_1) \times 1)
\]
\[ B (u^c_1) = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{k_1 1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} b^c_1 \\ 0 \end{bmatrix} ((k_1 \times 1) \quad ((k - k_1) \times 1)) \]
Identification – 6

Start from

\[
\Pi = \frac{-B \Gamma^{-1}}{Data \text{ Unknowns}}
\]

\[
\Pi \Gamma = -B
\]

\[
\Pi (\Gamma u_i^c) = -Bu_i^c
\]

\[
\Pi \begin{bmatrix} \gamma_1^c \\ 0 \end{bmatrix} = -\begin{bmatrix} b_1^c \\ 0 \end{bmatrix}
\]

\[
\begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} \begin{bmatrix} \gamma_1^c \\ 0 \end{bmatrix} = -\begin{bmatrix} b_1^c \\ 0 \end{bmatrix}
\]

\[
\Pi_1 \gamma_1^c = -b_1^c
\]

\[
\Pi_3 \gamma_1^c = 0
\]
\[ \Pi : (k \times g) \]
\[ \Pi_1 : (k_1 \times g_1) \]
\[ \Pi_2 : (k_1 \times (g - g_1)) \]
\[ \Pi_3 : ((k - k_1) \times g_1) \]
\[ \Pi_4 : ((k - k_1) \times (g - g_1)) \]

\[ \Pi_1 \gamma_1^c = -b_1^c \]
\[ \Pi_3 \gamma_1^c = 0 \]

Recursive system that can be solved starting from the second block.
Identification – 8

We find a solution for the second system and we insert it into the first one. The second system is homogeneous, we have the trivial solution $\gamma_1^c = 0$, and if it admits one solution for $\gamma_1^c \neq 0$ then we have an infinite number of solutions, that differ only for a scalar. This depends on the rank of $\Pi_3$. The necessary and sufficient condition to have an infinite number of solution for $\Pi_3 \gamma_1^c = 0$ is the Rank Condition:

$$\text{rank} (\Pi_3) = g_1 - 1$$

necessary and sufficient condition for the first equation to be identified.

Problem: In practice $\Pi_3$ has to be estimated. $\hat{\Pi}_3$ has with probability one rank equal to $g_1$. The only solution is vector $\gamma_1^c = 0$. With full rank we cannot identify the structural parameters.
Identification – 8

We adopt a weaker condition. Order condition, necessary condition

\[ k - k_1 \geq g_1 - 1 \]

If this inequality is satisfied then we can have identification otherwise it is sure that we cannot identify the structural parameters. In order to have

\[ \text{rank (}\Pi_3\text{)} = g_1 - 1 \]

it is necessary that the rows are in number larger than \((g_1 - 1)\).

If \( k - k_1 = g_1 - 1 \) the equation is exactly identified.

If \( k - k_1 > g_1 - 1 \) the equation is overidentified.
Identification – 9
The system is identified if all equations are exactly identified or overidentified. We must have

\[ k - k_i \geq g_i - 1 \quad i = 1, \ldots, g \]