Examples of stochastic processes

Eduardo Rossi
University of Pavia
Dependent processes Let \( \{x_t : t = 1, 2, \ldots \} \) be a scalar process defined on the positive integers.

Essentially stationary: \( E(x_t^2) \) is uniformly bounded.

An immediate implication is

\[
\sigma_T^2 \equiv Var(\sum_{t=1}^{T} x_t)
\]

is well defined.
If, in addition

\[ \sigma_T^2 = O(T) \]  \hspace{1cm} (1)

This condition implies that the variance of the partial sum is bounded above by a multiple of \( T \); it rules out highly dependent processes with positive autocorrelation that do not die out to zero sufficiently quickly.
\[ \sigma_T^{-2} = O(T^{-1}) \]  \hspace{1cm} (2)

This condition implies that the variance of the partial sum is bounded below by a (positive) multiple of \( T \); it rules out processes with strong negative serial correlation.

\[ \sigma_T^{-1} \sum_{t=1}^{T} (x_t - E(x_t)) \xrightarrow{d} N(0, 1) \]  \hspace{1cm} (3)

This condition states that \( \{x_t\} \) satisfies the CLT.

we say that \( \{x_t\} \) is weakly dependent.
Essential stationarity rules out strictly stationary processes that have an infinite second moment. This because stationary processes with an infinite second moment don’t satisfy the CLT.

There are processes exhibiting very little temporal dependence that violate (2).

Example

\[ x_t \equiv \epsilon_t - \epsilon_{t-1} \]

where \( \{\epsilon_t : t = 1, 2, \ldots\} \) is an i.i.d. sequence with finite second moment. Then \( \sigma_T^2/T \rightarrow 0 \) even though \( x_t \) and \( x_{t+j} \) are independent for \( j \geq 2 \). But the sequence does not satisfy the CLT.
It is the score of the objective function evaluated at the "true" parameters that should satisfy the CLT in applications with essentially stationary, weakly dependent data. For many problems this follows from the essential stationarity and weak dependence of the underlying process, along with some additional moment conditions.
Most of the early work on limiting distribution theory (focus on ml estimation) relied on the CLT for martingale difference sequence under correct dynamic specification.

Recent work covers a broader class of estimators and allows for dynamic misspecification. For many problems with misspecified dynamics the martingale CLT cannot be applied.

The econometric work on limit theory for estimation with essentially stationary, weakly dependent processes has relied on various mixing conditions available in the theoretical time series literature.
Under certain moment and mixing assumptions the process \( \{x_t\} \) satisfies the CLT.

In the strictly stationary case:

- Rosenblatt (1956) proves a CLT for \( \alpha \)-mixing (strong mixing) sequences.
- Billingsley (1968) proves results for \( \phi \)-mixing (uniform mixing) sequences and functions of \( \phi \)-mixing sequences.
- McLeish (1975) extended Billingsley’s results to allow for bounded heterogeneity.
• Wooldridge and White (1989) and Davidson (1992) have proven CLT’s for ”near epoch-dependent” (NED) functions of underlying mixing sequences.
Example Weakly dependent process. Let \( \{\epsilon_t : t = 1, 2, \ldots\} \) be an i.i.d. sequence with

\[
\sigma_{\epsilon}^2 \equiv E(\epsilon_t^2) < \infty
\]

\[
E(\epsilon_t) = 0.
\]

Let \( \{\phi_j : j = 0, 1, 2, \ldots\} \) be a sequence of real constants such that

\[
\sum_{j=0}^{\infty} |\phi_j| < \infty \tag{4}
\]

then we can define a process \( \{x_t : t = 1, 2, \ldots\} \) by

\[
x_t = \sum_{j=0}^{\infty} \phi_j \epsilon_{t-j} \quad t = 1, 2, \ldots
\]

i.e. \( \sum_{j=0}^{\infty} \phi_j \epsilon_{t-j} \) exists a.s.
Provided that

$$\sum_{j=0}^{T} \phi_j \neq 0$$

holds and it follows that

$$\sigma_0^{-1} T^{-1/2} \sum_{t=1}^{T} x_t \overset{d}{\rightarrow} N(0, 1)$$

where

$$\sigma_0^2 \equiv \sigma_\epsilon^2 \left( \sum_{j=0}^{\infty} \phi_j \right)^2$$
The summability condition on \( \{\phi_j : j = 0, 1, 2, \ldots\} \) ensures (1); it allows much more dependence than a stable ARMA process with i.i.d. innovations, but (4) does imply that \( \phi_j \to 0 \) as \( j \to \infty \) at a sufficiently fast rate.

We can allow for bounded heterogeneity by changing the assumption about the underlying \( \{\epsilon_t\} \): \( \{\epsilon_t\} \) independent not identically distributed, i.n.i.d., with

\[
E(|\epsilon_t|^{2+\delta})
\]

bounded for some \( \delta < 0 \).
Then Fuller (1976, theorem 6.3.4) implies that CLT holds (3).

When we relax the requirement that $E(x_t^2)$ is uniformly bounded we arrive at the notion of a \textit{globally nonstationary} process. Even though such processes are growing or shrinking over time, it is entirely possible for them to satisfy the CLT.

As an example of globally nonstationary but weakly dependent process, define

$$x_t = tu_t$$

where $\{u_t : t = 1, 2, \ldots\}$ is a weakly dependent series with $E(u_t^2)$ uniformly bounded and $E(u_t) = 0$, $t = 1, 2, \ldots$ (for example, $\{u_t\}$ could be i.i.d.).
Note that

\[ E(x_t^2) = O(t^2) \]
\[ \sigma_T^2 = O(T^3) \]

Processes that are strictly stationary and ergodic, that are not weakly dependent \((strongly dependent, \text{ Robinson (1991)})\). A general class of strongly dependent processes is given by

\[ x_t = \sum_{j=0}^{\infty} \phi_j \epsilon_{t-j} \quad t = 1, 2, \ldots \]

where the the coefficients \(\{\phi_j\}\) are square summable

\[ \sum_{j=0}^{\infty} \phi_j^2 < \infty. \]
Even though such a process is covariance stationary, without further constraints on \( \{ \phi_j \} \) the variance of the partial sum can be of order larger than \( T \) so (1) does not hold.

Examples are long memory or fractionally integrated processes with degree of integration between 0 and 1.

Little is known about the asymptotic distribution of estimators from general nonlinear problems when the underlying sequence is strongly dependent.
Nonergodic processes: processes that exhibit such strong dependence they do not satisfy the law of large numbers.

Example

\[ x_t = x_{t-1} + \epsilon_t \quad t = 1, 2, \ldots \]

where \( \{\epsilon_t : t = 1, 2, \ldots\} \) is an i.i.d. sequence and \( x_0 \) is a given random variable. Assume

\[ E(\epsilon) = 0 \]
\[ E(\epsilon^2) < \infty \]

Under these assumptions the first moment of \( x_t \) need not exist.
If we add
\[ E(x_0^2) < \infty \]
\(x_0\) is uncorrelated with all \(\epsilon_t\), then
\[
\begin{align*}
\text{Var}(x_t) &= O(t) \\
E(x_t) &= E(x_0) \quad \forall t
\end{align*}
\]
But the process \(\{x_t\}\) does not return to its mean with any regularity (nonergodic), and the sample average \(\bar{x}_T\) will not converge in probability to \(E(x_0)\).