Open Market Operations, Interbank Market and Over-collateralization

G. Ferrero, M. Loberto, M. Miccoli *

FIRST DRAFT
January 29, 2013

Abstract

This paper provides a micro-founded general equilibrium description of interbank markets and analyzes positive implications of the effect of central bank’s open market operations on prices and quantities exchanged on the interbank market. First a model with only nominal and risk free government bonds is presented: in this setup open market operations that alter the composition of the central bank’s balance sheet can affect quantities exchanged on the interbank market even if the economy is in a liquidity trap equilibrium. Then a real asset is introduced in order to determine the effects on prices and quantities of different unconventional monetary policies that alter the dimension of the central bank’s balance sheet: a swap of real asset for money and a swap of real asset for bonds. Finally, the effect of volatility shocks on the efficacy of Central Bank’s unconventional monetary policies is discussed and some preliminary evidence is provided on the empirical validity of the model.

1 Introduction

During the recent financial crisis the banking sector has been characterized by high instability. A rising in counter-party risk perception and a lowering of the quality of collateral fostered increased haircuts and a thinning interbank market. Central Banks, in order to provide liquidity and ease tensions in the interbank market, undertook unconventional monetary policy operations aimed at substituting distressed asset with reserves. These operations, even though they eased tensions, did not increase volumes exchanged on the interbank market. The following quote by the European repo market survey of ICMA exemplifies the point:

The sharp contraction in the survey total [of repo contracts value] to EUR 5,647 billion from EUR 6,204 billion in December 2011 (-9.9% when adjusted for survey sample changes) reflects a number of factors: [...] the impact of the

---

*Banca d’Italia, Via Nazionale 91, 00184 Roma, Italy. Corresponding author: marcello.miccoli@bancaditalia.it. We thank participants at the Banca d’Italia lunch seminar for comments. The views expressed here are those of the authors and do not necessarily represent in any way the views of the Bank of Italy.
two massive LTRO by the ECB in December 2011 and February 2012 (both after the last survey). These refinancings have helped to calm the markets but, at the same time, have reduced the need of many banks to use the markets to manage their liquidity.\footnote{ICMA, European repo market survey, Number 23, August 2012}

More in general in the euro area, after the beginning of the sovereign crisis, we have observed that volumes exchanged on collateralized interbank exchanges became negatively correlated with liquidity provisions operations of the ECB. Figure 1 shows the time series for the total volume of Repo in Europe and the liquidity introduced through the MRO and LTRO operations of the ECB.

How providing the market with liquidity influences the interbank market? Can we rationalize the time series evolution of liquidity and Repo funding in figure 1? If the answers to these question is positive, can we use this model to predict the effect on the interbank market of unconventional monetary policies? Our work is aimed at answering these questions by building a general equilibrium micro-founded model of the interbank market in which open market operations influence the banks’ incentive to lend to each other. The general equilibrium analysis of Central Bank’s operations on financial intermediation is the main contribution of our work and only recently its features have received attention in the literature. Moreover, the predictions of the effects of unconventional monetary policies on the interbank market is a paramount policy question, given the large use that central
banks in the developed economies made of these operations during the financial crisis.

The model is based on the *New Monetarist Economics*\(^2\) literature stemming from Lagos and Wright (2005) in which money is an essential medium of exchange, that is, in its absence the economy converges to a Pareto inferior equilibrium. The basic idea is that in order to trade in decentralized markets characterized with limited commitment and no record keeping technology a commonly accepted medium of exchange is needed: money fulfills this role.

In the first part of our work we build a model where money and a risk free bond are issued by a consolidated government that controls the composition of its balance sheet, and therefore, the relative scarcity in the market of money and bonds through open market operations. Consumers are uncertain whether they will face markets in which goods can be traded only for money or for claims on other assets. Banks fill in this missing market by collecting deposits and providing their depositors, once uncertainty is realized, with their needed medium of exchange.

Differently from other works in the literature of financial intermediation that consider partial equilibrium analysis à la Diamond and Dybvig (1983), we build on this general equilibrium model a specific role for reserves transfer between banks. In particular we assume that banks face an idiosyncratic money demand shock: they are uncertain about the composition of their depositors. This gives rise to an interbank market in which banks lend reserves to each other subject to a collateral requirement. The advantage of the general equilibrium framework is that it allows to derive monetary policy implications which are not limited to the interbank market, but span all endogenous object in the economy, in particular monetary policy decisions also influence equilibrium asset prices, consumption and production decisions.

As customary in this literature we focus our analysis on stationary equilibria where real variables and inflation are constant over time. In the equilibrium the price of the bond can feature a liquidity premium, that is, the premium bonds command by being able to facilitate transactions. By increasing (decreasing) nominal money balances relative to bonds the monetary authority influences the equilibrium price of the bond and, therefore the degree of substitutability between money and bonds and the terms at which banks lend to each other. Hence monetary policy decisions determine the long run equilibrium of the economy. Similarly to Williamson (2012) our model features a liquidity trap equilibrium out of the Friedman rule, in which the real returns on currency and interest-bearing bonds are equal (the nominal interest rate is zero). Differently from this work, however, in our liquidity trap equilibrium open market operations are not always irrelevant, since they affect lending quantities in the interbank market (even though they do not change equilibrium consumption). In particular open market operations that swap government bonds for reserves reduce the demand of reserves on the interbank market, and therefore

---

\(^2\)See for instance the treatment in Williamson and Wright (2010), Williamson and Wright (2011).
the equilibrium volumes exchanged. The model is therefore consistent with the available data provided in figure 1.

In the second part of our work we introduce in this economy a real asset in positive net supply with a stochastic one-period dividend. This asset can be exchanged by agents (and banks) and can be posted as collateral on the interbank market, however the realization of its dividend is public knowledge before interbank loans settlement. The assumption of the absence of a commitment technology for banks implies that banks cannot commit to repay loans higher than the nominal amount corresponding to the lowest possible realization of the asset posted as collateral. This gives rise to over-collateralization on the asset in the interbank market, since its pledgeable value is lower than its expected return. In this setup also the price of the asset can feature a liquidity premium, which depends on monetary policy decisions. The main effect of introducing the real asset with respect to the economy with only money and bonds is that it alters the regions of parameters that define the equilibria of the economy. In particular the presence of the asset, by increasing the collateral pool available to banks, decreases the probability for the economy of experiencing a liquidity trap equilibrium.

Within this framework we then analyze the effect of different unconventional monetary policies. The issue of how open market operations that change the size of the central bank balance sheet affects allocations and prices in the economy when conventional monetary policy is constrained by the zero lower bound is particularly relevant in the current policy debate. However our model can also be used to address another feature of unconventional monetary policy. The recent measures adopted by some central banks explicitly involve open market operations that swap risky asset for reserves can be analyzed in our framework. By increasing reserves available to the banking sector and reducing the available collateral in the economy the Central Bank increases its liquidity premium, implying a lower cost on the interbank market, and therefore an increase in the probability of the liquidity trap.

In the last section of the work we also provide two other different types of analysis. First we analyze the effect of open market operations that swaps assets for government bonds. Since there is over-collateralization, the operation increases the pledgeable amount of assets in the economy and therefore fosters an increased demand of reserves on the interbank market and a rise in the rates charged. The second analysis that we provide is how an increase in uncertainty about the future dividend of the real asset influences the equilibrium. Our characterization of the collateral requirement has the advantage of providing a straightforward characterization of equilibrium prices and real quantities exchanged on the interbank market when uncertainty changes. An increase in uncertainty (a mean preserving spread of asset’s returns) increases haircuts and reduces demand for reserves on the interbank market if the asset is not sufficiently abundant. However, a reduction of uncertainty when the economy is in the liquidity trap, by decreasing the
over-collateralization, provides incentives for banks to increase the demand of liquidity in
the interbank market; this, in turn, increases interest rates and foster an exit from the
liquidity trap equilibrium for the economy.

The structure of the work is the following. Section 1.1 discusses the closest relevant
papers, while section 2 provides a description of the model economy with only money
and governments bonds. Section 2.5 solves the problem for this economy and briefly
presents the different types of equilibria. Section 3 then introduces the risky asset in the
economy, solves for the equilibria in this economy and explores the effect of different types
of unconventional monetary policies. Section 4 provide some suggestive evidence on the
empirical validity of our model and section 5 concludes.

1.1 Related Literature

Williamson (2012) analyzes monetary policy in a general equilibrium model where money
is micro-founded à la Lagos and Wright and financial intermediaries emerge naturally to
insure agents against idiosyncratic liquidity shock (as in Diamond and Dybvig (1983)).
Central bank can change the composition of public liquidity (currency and government
bonds) throughout open market operations and a liquidity trap equilibrium, where the
nominal interest rate is zero and banks keep excesses reserves, can emerge for any long-
run money growth rate, whenever currency is plentiful relative to government bonds. We
build our model on Williamson (2012), but we consider the case in which also banks faces
idiosyncratic liquidity needs and a secured interbank markets allows the redistribution of
liquidity among banks.

Bhattacharya and Gale (1987) were the first to consider idiosyncratic liquidity demand
for banks in a Diamond and Dybvig environment. Recently, many authors have studied
interbank markets, trying to explain the market freeze followed the 2007 financial crisis.
Among others Allen, Carletti and Gale (2009) shows that when banks face both idiosyn-
cratic and aggregate liquidity shocks, interbank market is characterized by excess price
volatility and a central bank, through open market operations, can stabilise the interbank
interest rate. Differently from them, in our model banks face only idiosyncratic liquidity
shocks. Also Freixas, Martin and Skeie (2011) suggest that a central bank should fix the
interest rate the interbank market, where, otherwise, the possibility of a financial crisis
generates equilibrium indeterminacy. While all these models follow essentially a partial
equilibrium analysis, we embed an interbank market in a general equilibrium model à la
Lagos and Wright, where central bank operations affect not only the interbank market,
but also all other quantities and prices.

Gertler and Kiyotaki (2011) introduces interbank markets in a classical real business
cycle model and show how disruptions in financial intermediation can induce a crisis that
affects real activity. Moreover, they analyze the effects of unconventional monetary policy,
as direct lending from the central bank to the private sector and equity injections in bank’s
capital. We consider, instead, open market purchases of government bonds and real assets from the central bank and the effects on the long run equilibrium. Similarly to Kiyotaki and Moore (2012), in our model the efficacy of open market operations that involve the purchase of real assets relies on their partial pledgeability. However, differently from them, the effects of real assets purchases are different if the central bank sells government bonds or injects fiat money.

Curdia and Woodford (2011) extend a standard New Keynesian model to allow a role for central bank’s balance sheet in equilibrium determination. In their model, however, assets do not have any medium-of-exchange role and the only unconventional policy that can be effective is direct lending to the private sector. Instead, taking into account micro-foundations, we show that simple open market operations, affecting liquidity premia of the assets, can have real effects not only on the short run, but also on the long run.

2 Model with only money and bonds

Time is infinite and discrete. As in Lagos and Wright (2005), each period is divided into two subperiods. In the first subperiod (day) agents trade in a decentralized market (DM), while in the second subperiod (night) they trade in a walrasian market (centralized market, CM).

There are two types of economic agents: buyers and sellers. There is a continuum with mass 1 of buyers, and each buyer has preferences given by

\[ E \sum_{t=0}^{\infty} \beta^t [u(q_t) - h_t] \] (1)

where \( q \) is the consumption of good 1, produced by the seller during the day and not storable, while \( h \) is the difference between labor supply and consumption of a not storable good 2, produced by the buyer during the night with a linear technology. We assume that \( u(\cdot) \) is a strictly concave continuous and differentiable function, \( u'(\cdot) > 0, u''(\cdot) > 0, \) and satisfies the Inada conditions, \( u'(0) = \infty \) and \( u'(\infty) = 0. \)

There is a continuum with mass 1 of sellers, and each seller has preferences given by

\[ E \sum_{t=0}^{\infty} \beta^t [-q_t + c_t] \] (2)

where \( q \) is the disutility to produce \( q \) units good 1, and \( c \) is the consumption of good 2. Define \( q^* \) as the optimal amount of consumption in the DM, given by the solution to \( u'(q^*) = 1. \)

In the DM, buyers and the seller meet randomly and trade pairwise. The terms of the trade are determined through a Nash Bargaining Process in which, for analytical
convenience, all the bargaining power is given to the buyer. During the CM, instead, buyers and sellers trade in a walrasian market.

In a frictionless environment, where unsecured credit is feasible, sellers produce $q^*$ units of good 1 in the DM for buyers, receiving in exchange an equivalent amount of good 2 during the CM market. However, introducing limited commitment and assuming there is no record-keeping technology unsecured credit is anymore a feasible trading arrangement. In fact, the best strategy of buyers is to make always default and sellers, anticipating that, refuse in the DM any unsecured promise of buyers. Therefore, every trade in the DM must be *quid pro quo* because sellers want to exchange good 1 only for claims that can be exchanged for goods in the future. Indeed, there is a need for a medium-of-exchange.

In the first part of the paper, we consider the case in which there are only two assets: fiat money, $M$, and government bonds, $B$. Fiat money is a durable, tangible but intrinsically useless object in positive net supply in the private sector and injected by a central bank. Government bonds are intangible one period assets issued by the government at time $t$ at a price $\psi_t$ and pay one unit of fiat money at $t+1$. They represents account balances that agents hold with the government. All assets, money and bonds, are traded only in the CM.

We assume that in a fraction $\rho$ of meetings there is no record-keeping technology that allows sellers to signal default of buyers on credit arrangements. In this meetings there is a need for a tangible object that serves as medium-of-exchange, and the only object with this properties is fiat money. In a fraction $1-\rho$ of meetings there is no record-keeping, but a technology is available allowing sellers to verify the ownership by the buyers of claims on financial assets. Therefore, sellers accept to trade in secured credit arrangement, because they know that buyers have collateral that can be seized in the CM. It is possible to interpret our environment also in terms of differences of recognizability among different assets. While fiat money is universally recognized by all agents in the economy, only a fraction of them is able to recognize other assets, or, have the technology to verify the effective ownership of an asset.

Buyers discover in which type of meeting, and therefore which type of medium of exchange is necessary for the trade, at the beginning of the DM. Ex ante, each buyer will leave the CM with both money and government bonds. Then, in a fraction $\rho$ of meetings he has an amount of government bonds that are useless from exchange and

---

3 As Williamson (2012), we assume that the fraction of meetings in which only fiat money is accepted is fixed and exogenous. However, he assumes that in fractions $1-\rho$ of meetings there is a costless technology that allow buyers to transfer claims on financial intermediaries to sellers. Instead, we assume that in these meetings sellers are able to verify the effective ownership of an asset by buyers. Therefore, sellers give good 1 to buyers because they know that there is a collateral they can seize if in the CM buyers refuse to honor their promises.

4 The relation between recognizability and liquidity of different assets received recently a lot of attention in the literature. Among others, Lester, Postlewaite and Wright (2012) show that the fraction of sellers who accept an asset can be endogeneized, provided they have to invest in costly information or technology that allow them to recognize the asset.
this reduces his potential consumption. Instead, in a fraction $1 - \rho$ he can use all the asset in his portfolio, but since money is generally dominated in return by government bonds he would be better off with a portfolio of only bonds. This give rise to risk-sharing role for financial intermediaries as in Diamond and Dybvig (1983), who through the diversification over different buyers rules out idiosyncratic liquidity shocks and offer Pareto-superior allocations.

In the CM all sellers, buyers and the government meet in a centralized Walrasian market where market for bonds, money and good 2 clear, given the choices of consumption and production of the agents.

2.1 Financial Intermediaries

There is a continuum of banks with mass one. Banks accept good 2 as deposit from buyers in the CM of time $t$ and invest deposits in money and bonds. In the DM, at $t + 1$, banks allow their depositors to withdraw a predetermined amount of money or claims on their deposit, that can be used as medium of exchange, depending on the type of meeting they face. A fraction of sellers accept claims on deposits because they recognize they are backed by assets held by the banks in their balance sheets. At the beginning of the CM, these sellers go the banks and cash those claims.

Banks therefore offer a liquidity insurance to the buyers. Any remaining money and asset are then redistributed to its depositors in the CM at $t + 1$, at which point banks are liquidated and a new generation of banks is created. Since the banking sector is perfectly competitive, banks offer deposit contracts that maximize the expected utility of the buyer and earn zero profits.\footnote{Note that only buyers require liquidity services, since sellers do not consume in the decentralized market. As will be clear later, this also implies that sellers never have the convenience to demand fiat money in the CM out of the Friedman rule, because the marginal cost is greater than the marginal benefit. The same is true when price of government bonds is above his fundamental value because incorporates a liquidity premium. They would be indifferent if the price of bonds reflects only the fundamental value, but in this case, without loss of generality, we assume they do not demand any amount of bonds.}

Differently from Williamson (2012), we assume that banks cannot obtain perfect diversification with respect to the types of buyer that have as depositors. With probability $1/2$ at time $t + 1$ a bank has a fraction $\rho + \varepsilon$ ($0 < \varepsilon < 1 - \rho$ and $\varepsilon < \rho$) of buyers who can buy only with money and $1 - \rho - \varepsilon$ that can use also bonds. We denote them as type 1 banks. With probability $1/2$, instead, a bank has a fraction $\rho - \varepsilon$ of buyers who can buy only with money and $1 - \rho + \varepsilon$ that can use also bonds. We denote them as type 2 banks. They discover their type at the beginning of the DM.

This gives rise to the need for liquidity transfers among banks, the interbank market. Given the continuum hypothesis, half the fraction of the banks will face $\rho - \varepsilon$ depositors that require money and can transfer excess liquidity to the remaining banks that face $\rho + \varepsilon$ depositors that require cash. In the interbank market bank collateral is required.
particular, for a given amount of nominal bond $B$, a bank will transfer not more than $RL$, where $L$ is the nominal value of money transferred and $R$ is the gross nominal interest rate on interbank market. Interbank loans are settled at the beginning of the CM\(^6\).

2.2 Consolidated government

We consider central bank and government as a consolidated entity. At time $t$, in the CM government injects an amount of money $M_t$ and issue an amount $B_t$ of one-period government bonds. Each unit of bond is issued at a price $\psi_t$ and pays one unit of fiat money in the CM at time $t+1$. We assume that government levy lump-sum taxes $T_t$ on buyers in the CM. Taxes are denominated in terms of goods 2.

Letting $\phi_t$ denote the price of money in terms of goods in the CM, the consolidated government budget constraint is

$$\phi_t(M_t + \psi_t B_t) + T_t = \phi_t(M_{t-1} + B_{t-1}).$$

Monetary authority commits to a policy such that the total stock of nominal government liabilities, $M + B$, grows at a constant gross rate $\gamma$. Moreover, as in Williamson (2012) the government keeps the ratio of currency to the total nominal government debt, $\delta$, constant:

$$M_t = \delta(M_t + B_t).$$

Here, $B_t$ denotes the bonds held by the private sector. We consider $B_t \geq 0$ for all $t$ (the government is a net debtor), that it is equivalent to restrict $\delta$ in the interval $(0, 1]$\(^7\). We assume that the government starts in period zero with no outstanding liabilities and that fiscal policy is purely passive: the path of lump-sum taxes changes passively to support chosen paths for the nominal liabilities of the government.

2.3 Centralized Market

As standard in this literature, we will focus our analysis to stationary equilibria where $\frac{\phi_t}{\phi_{t+1}} = \gamma$ and all prices and real variables are constant (the exact definition of the equilibrium is given in the next section), therefore we will omit time indexes from now on.

Since the banks are perfectly competitive, they will maximize the utility of their depositors. In the CM each bank, using the deposits of buyers, has to choose how much

\(^{6}\)The introduction of a collateral requirement makes a reader asking itself if banks have limited commitment and how can be possible they can write a contract with depositors if they can renge their promises. When we analyze the full model with a risky assets we discuss this issue. Here, given banks have a predetermined budget, if they have full commitment the problem is equivalent. When they ask for an interbank loan $L$ they still need to set apart an amount of bonds $\frac{B}{R}$ for the repayment. Therefore, it is like the loan is collateralized.

\(^{7}\)We restrict the support of $\delta$ because we want to concentrate our analysis on a more realistic situation in which the government is a debtor. Williamson (2012) consider the general case in which $\delta \in (-\infty, \infty)$.  

9
real money holdings and bonds carry over to the next period, where they will be used by buyers as medium-of-exchange in the DM. If a bank wants a real amount of money \( m \) in the next DM he needs to invest \( \gamma m \) in the previous CM (the same holds for bonds). That is because, by letting \( \hat{m} \) be the nominal amount of fiat money the bank holds in the CM, the real amount of money next period will depend on the price level \( \phi_{t+1} \), specifically \( m = \phi_{t+1} \hat{m} \). Since the real money holding in period \( t \) are given by \( \phi_t \hat{m} \), we have that

\[
\phi_t \hat{m} = \frac{\phi_t}{\phi_{t+1}} \phi_{t+1} \hat{m} = \frac{\phi_t}{\phi_{t+1}} \hat{m} = \gamma m.
\]

The same logic applies to the real value of bonds holdings \( b \).

The bank’s problem in the CM is given by:

\[
\max_{m,b} -\gamma m - \psi \gamma b + \beta E V(m,b)
\]

where \( E V(m,b) \) is the value function associated with having real money holdings \( m \) and bonds \( b \) in the DM of the next period\(^8\). Since the banks are perfectly competitive, they will maximize the utility of the buyer, and the first order conditions of this problem are:

\[
[m] : \quad \frac{\gamma}{\beta} = E V_m(m,b), \quad (6)
\]

\[
[b] : \quad \psi \frac{\gamma}{\beta} = E V_b(m,b). \quad (7)
\]

### 2.4 Decentralized Market

In the DM, each bank will choose how much to transfer to its depositors given the possibility of accessing the interbank market. Given that in the DM uncertainty about the identity of its depositors is resolved, the problem of the bank will be different depending

---

\(^8\)This value function will be explicitly defined once we discuss the problem of the DM.
on the realization of uncertainty. Here we define the maximization problem of a bank (called type 1) that is cash constrained, that is it faces $\rho + \varepsilon$ depositors that require only cash for their transaction. As such it will have an incentive to access the interbank market asking for liquidity to the other fraction of banks that face $\rho - \varepsilon$ depositors needing cash.

Type 1 bank will maximize the utility of their depositors knowing that a fraction $\rho + \varepsilon$ of them will require cash, while a fraction $1 - \rho - \varepsilon$ can accept claims on bank deposits. The problem of type 1 banks is:

$$\max_{\hat{m}_1, \hat{b}_1, l} (\rho + \varepsilon) u\left(\frac{\hat{m}_1 + l}{\rho + \varepsilon}\right) + (1 - \rho - \varepsilon) u\left(\frac{b - \hat{b}_1 + m - \hat{m}_1}{1 - \rho - \varepsilon}\right) + (\hat{b}_1 - Rl)$$

s.t.

$$\hat{b}_1 \geq 0$$ (9)

$$\hat{m}_1 \leq m$$ (10)

$$l \geq 0$$ (11)

$$Rl \leq \hat{b}_1$$ (12)

where $\hat{m}_1$ is the amount of cash given to depositors who are in cash only exchanges, $\hat{b}_1$ is the nominal amount of asset used by the bank as collateral on the interbank market (and so $m - \hat{m}_1$ and $b - \hat{b}_1$ represents respectively the (real) amount of cash and bonds given to buyers in claims meetings) and $l$ is the (real) amount of money obtained on the interbank market. The last term in the objective function defines the possibility for the bank to accumulate excess reserves that will be redistributed to all depositors in the CM9. Two remarks: i) since we are assuming that buyers make a take-it-or-leave-it offer to sellers and they have a linear disutility from producing good 1, the quantity consumed by buyers equals their purchasing power; ii) variables are expressed in aggregate quantities, so that each depositor will receive a share of $\frac{1}{\rho + \varepsilon}$ or $\frac{1}{1 - \rho - \varepsilon}$ of the disbursed amount. Therefore we can replace in the utility function quantities consumed of good 1 with the real value of money and claims on deposits of buyers.

The constraint on the interbank market exchanges is $Rl \leq \hat{b}_1$: in order to receive money $l$ the bank will have to post collateral of at least $\hat{b}_1/R$, where $R$ is the gross interest rate on interbank exchanges. The first order conditions of the problem are

$$[\hat{m}_1] : \quad u'\left(\frac{\hat{m}_1 + l}{\rho + \varepsilon}\right) - u'\left(\frac{b - \hat{b}_1 + m - \hat{m}_1}{1 - \rho - \varepsilon}\right) - \lambda^1 = 0$$

$$[\hat{b}_1] : \quad -u'\left(\frac{b - \hat{b}_1 + m - \hat{m}_1}{1 - \rho - \varepsilon}\right) + 1 + \mu^1 + \zeta = 0$$

$$[l] : \quad u'\left(\frac{\hat{m}_1 + l}{\rho + \varepsilon}\right) - R - R\zeta + \nu^1 = 0$$

9Utility of agents is linear in the consumption good of the CM market, hence the linearity of this term.
where $\lambda^1$ is the Lagrange multiplier associated with (10), $\mu^1$ is associated with (9), $\nu^1$ with (11) and $\zeta$ with (12).

The problem of the banks that face $\rho - \varepsilon$ depositors who require cash in their exchange meeting (type 2 bank) is specular to the previous case and is defined by:

$$\max_{\hat{m}^2, \hat{b}^2, d} \ (\rho - \varepsilon)u\left(\frac{\hat{m}^2}{\rho - \varepsilon}\right) + (1 - \rho + \varepsilon)u\left(\frac{b - \hat{b}^2 + m - \hat{m}^2 - d + Rd}{1 - \rho + \varepsilon}\right) + \hat{b}^2$$

s.t. $\hat{b}^2 \geq 0$ (17)

$$\hat{m}^2 + d \leq m$$ (18)

$$d \geq 0$$ (19)

where $\hat{m}^2$ is the amount of money given to its depositors in cash only meetings, $\hat{b}^2$ is the nominal amount of assets that the bank chooses to keep as excess reserves and redistribute in the CM, and $d$ is the loan on the interbank market. Note that $(R - 1)d$ represent the net return from operating on the interbank market, and as such is a resource that can be given to the depositors. The first order conditions are:

$$[\hat{m}^2] : \ u'\left(\frac{\hat{m}^2}{\rho - \varepsilon}\right) - u'\left(\frac{b - \hat{b}^2 + m - \hat{m}^2 - d + Rd}{1 - \rho + \varepsilon}\right) - \lambda^2 = 0 \quad (20)$$

$$[\hat{b}^2] : \ -u'\left(\frac{b - \hat{b}^2 + m - \hat{m}^2 - d + Rd}{1 - \rho + \varepsilon}\right) + 1 + \mu^2 = 0 \quad (21)$$

$$[d] : \ (R - 1)u'\left(\frac{b - \hat{b}^2 + m - \hat{m}^2 - d + Rd}{1 - \rho + \varepsilon}\right) - \lambda^2 + \nu^2 = 0 \quad (22)$$

where $\lambda^2$ is the Lagrange multiplier associated with (18), $\mu^2$ is associated with (17) and $\nu^2$ with (19).

### 2.5 Equilibria of model with only money and bonds

As we have anticipated, we concentrate on stationary equilibria where $\frac{\phi_t}{\phi_{t+1}} = \gamma$ and real variables are constant. The definition of equilibrium is the following:

**Definition 1 (Equilibrium Definition)** Given a monetary policy rule $(\gamma, \delta)$, a stationary equilibrium consists of real quantities of currency $m$ and interest bearing asset $b$, bank transfers $\hat{m}^i, \hat{b}^i$ for each bank type $i = 1, 2$ and interbank loans $l$ and $d$ such that, for given an initial tax $T_0$ and tax rate $T_t$ for $t = 1, 2, \ldots$, a gross interest rate on interbank market $R$ and a bond price $\psi$, \{m, b, \hat{m}^i, \hat{b}^i, l\} i) solve problems (5), (8) and (16) when $\frac{\phi_t}{\phi_{t+1}} = \gamma$, ii) prices are such that all markets clear ($l = d$ and $b = m(1/\delta - 1)$), iii) deposit contracts do not depend on bank’s type, iv) the government budget constraint (3) holds.
In order to simplify notation it is useful to define the following quantities:

\[ q_{11} = \frac{\tilde{m}^1 + l}{\rho + \varepsilon}; \quad q_{21} = \frac{b - \tilde{b}^1 + m - \tilde{m}^1}{1 - \rho - \varepsilon}; \]
\[ q_{12} = \frac{\tilde{m}^2}{\rho - \varepsilon}; \quad q_{22} = \frac{b - \tilde{b}^2 + m - \tilde{m}^2 - d + Rd}{1 - \rho + \varepsilon}, \]  

where \( q_{ij} \) is the quantity consumed by a buyer in the DM. \( i = 1 \) if the buyer is in a cash only meeting and \( i = 2 \) otherwise. While \( j = 1 \) if the buyer deposits in a bank with higher than average depositors in a cash meeting and \( j = 2 \) otherwise. We focus on standard deposit contracts. This implies that every depositor has the right to withdraw a fixed amount of real money in the DM independently of the type of bank he deposits to. Therefore we concentrate on equilibria where

\[ q_{11} = q_{12} \text{ and } q_{21} = q_{22}. \]  

Having defined the problem for each type of bank in the decentralized market, we can explicitly write down the value function \( V(m, b) \) of the centralized market problem. Given that ex-ante each bank will have half the probability of being one of the two types, the value function is given by:

\[ EV(m, b) = \frac{1}{2} \left[ (\rho + \varepsilon)u(q_{11}) + (1 - \rho - \varepsilon)u(q_{21}) + (\tilde{b}^1 - Rl) \right] + \frac{1}{2} \left[ (\rho - \varepsilon)u(q_{12}) + (1 - \rho + \varepsilon)u(q_{22}) + \tilde{b}^2 \right]. \]

Taking derivatives with respect to \( b \) and \( m \), and considering the constraints that the bank will face in the DM (10) and (18), the first order condition of the centralized market (6) and (7) can be expressed as:

\[ \frac{\gamma}{\beta} = \frac{1}{2} \left[ u'(q_{21}) + \lambda^1 \right] + \frac{1}{2} \left[ u'(q_{22}) + \lambda^2 \right], \]
\[ \frac{\psi}{\beta} = \frac{1}{2} \left[ u'(q_{21}) \right] + \frac{1}{2} \left[ u'(q_{22}) \right]. \]

These, together with first order conditions (13), (14), (15), (20), (21), (22), complementary slackness conditions, standard contract conditions, monetary policy definition (4) and equilibrium in the interbank market \( (l = d) \), define the system of equations the provides necessary and sufficient conditions for determining the equilibrium value of the unknowns of the problem, given by \( \{ m, b, \tilde{m}^1, \tilde{b}^1, \tilde{m}^2, \tilde{b}^2, l, d, \psi, R \} \). Note that taxes adjusts in order for the government budget constraint to hold.

The following propositions give a first characterization of the equilibria.

**Proposition 1 (Necessary and sufficient condition for equilibrium)** Necessary and
sufficient condition for an equilibrium to exist is $\gamma \geq \beta$.

**Proof.** Consider equation (25). The right hand side has to be at least one, since optimal consumption is defined by $u'(q^*) = 1$ and $\lambda^1, \lambda^2 \geq 0$. Therefore $\frac{\gamma}{\beta} \geq 1$ is both necessary and sufficient for the existence of an equilibrium. ■

The requirement of proposition 1 is equivalent to requiring that the nominal interest rate on bonds be weakly positive. Consider the Fisher equation $(1 + i) = (1 + r)(1 + \pi)$, where $1 + r$ is the gross real interest rate and $1 + \pi$ is the gross inflation rate. In this setup linearity in the CM’s utility implies that the real gross interest rate can never be higher the $1/\beta$ and gross inflation is given by $\gamma^{10}$. Therefore a weakly positive interest rate implies $\frac{1}{\beta} \gamma \geq 1$.

Given the initial restriction on parameters the following proposition states the possible bond prices in equilibrium.

**Proposition 2 (Equilibrium prices)** In every equilibrium $\frac{\beta}{\gamma} \leq \psi \leq 1$ and, whenever the interbank market exists, $R = \frac{1}{\psi}$.

**Proof.** In the appendix. ■

In equilibrium, from market clearing conditions both money and bonds must be held by agents. Therefore $\psi$ cannot be greater than one otherwise there would be no demand for the asset since at that point it would be better for agents to carry only money, and $\psi$ cannot be less than $\frac{\beta}{\gamma}$ otherwise there would be an infinite demand for bonds since they would more than compensate for inflation. The result that $R = \frac{1}{\psi}$ comes from a no-arbitrage condition: the return on the bond $\frac{1}{\psi}$ must equal to the return on money lent in the interbank market $R$. If for instance $\frac{1}{\psi} > R$ than a bank could make arbitrarily positive profits by buying bonds and using them to obtain money at a smaller cost.

Given proposition 1 and 2, depending on the ratio of currency on total nominal liabilities $\delta$ and growth rate of money $\gamma$, there will have different types of equilibrium, reflected in different steady state prices and consumption and real money holdings. In particular we will have:

1. **Plentiful government bonds case:** for $\gamma > \beta$ and $\delta < \hat{\delta}$ (where $\hat{\delta}$ depends on underlying parameters) then $\psi = \frac{\beta}{\gamma} = \frac{1}{R}$. In this equilibrium agents consume the optimal quantity if they are in a claims meeting, while they consume less than the optimal quantity in the cash only meeting, and the interbank market is active.

2. **Scarce government bonds case:** for $\gamma > \beta$ and $\hat{\delta} \leq \delta < \rho$ then $\psi = \frac{\delta}{1 - \rho} \frac{1 - \rho}{\rho} = \frac{1}{R}$.

In this equilibrium all buyers consume less than the optimal quantity, the interbank market is active and quantities exchanged on this do not change with respect to the plentiful government bonds case. However since now bonds are relatively more

---

10The price level in the model is $1/\phi_t$ so that gross inflation is $\frac{1}{1/\phi_{t+1}} = \gamma$
scarce the interest rate on the interbank market has to adjust with respect to the ratio of nominal bonds and money in the market, $\delta$.

3. **Liquidity trap with interbank market case**: for $\gamma > \beta$ and $\rho \leq \delta < \rho + \varepsilon$ then $\psi = 1 = \frac{1}{R}$. In this equilibrium buyers consume the same quantity irrespective if they are in a cash only meeting or not, and quantity consumed is less than the optimal quantity, the interbank market is active but the quantity exchanged decreases with $\delta$;

4. **Liquidity trap with no interbank market case**: for $\gamma > \beta$ and $\delta \geq \rho + \varepsilon$ then $\psi = 1$. In this equilibrium consumption is unchanged from previous case, however liquidity is so abundant that there is no role for the banking system anymore;

5. **Friedman Rule case**: for $\gamma = \beta$ and any value of $\delta$ then $\psi = 1$, consumption is optimal for any buyers and there is no role for the banking system.

In the following subsections we analyze each equilibrium in turn (leaving tough the formal derivation of each equilibrium to the appendix), providing necessary conditions on parameters for these equilibria to exists.

### 2.5.1 Plentiful government bonds equilibrium: $\psi = \frac{\beta}{\gamma}$ and $\gamma > \beta$

In this equilibrium the nominal interest rates on bonds is positive: it is given by $1/\psi - 1 = \gamma/\beta - 1 > 1$. However there is no liquidity premium on bonds, since the gross real interest rate on nominal bonds is $\phi \psi - 1 = \frac{1}{\beta}$, that is, it equals the rate of time preference. The fact that there is no liquidity premium indicates that bonds are not scarce. In this case buyers who are in a meeting exchange where claims on banks’ deposit are accepted, they will be able to consume the optimal quantity $q^*$ independently of the type of bank they are facing and without that the standard contract constraint is binding.

In this equilibrium since the price of the bond is so low, banks will have all the nominal bonds needed to issue claims to finance buyers, they will carry excess reserves to the next centralized market $\hat{b}^1 > 0$ and $\hat{b}^2 > 0$, and the constraint on the interbank market is slack $Rl < \hat{b}^1$.

Type 1 banks will go on the interbank market to obtain more cash for its depositors. However lending on the interbank market is not cheap, since it requires $R = \frac{\gamma}{\beta} > 1$, and depositors of type 1 bank in cash meeting only will not be able to consume the optimal quantity. But since the constraint on the interbank market is not binding, the marginal cost and benefit of having one more unity of money for both banks are equal, therefore also consumptions of depositors in cash only meetings across the two banks will be equalized. Therefore $q^{11} = q^{12} = \frac{m}{\rho} < q^*$ and the quantity exchanged on the interbank market is
\( l = \varepsilon \frac{\mu}{\rho} \), where from equation (25) the value of \( m \) is fixed by:

\[
\frac{\gamma}{\beta} = u'(\frac{m}{\rho})
\]

(27)

For this equilibrium to exists \( \delta \) must be low enough such that in the market there is plentiful of assets, or that \( \delta^2 > 0 \) and \( R_l < \delta^1 \). The two conditions imply that \( \delta < \delta^0 \) where

\[
\delta^0 \equiv \min \left\{ \frac{m}{(1 - \rho - \varepsilon)q^* + \left(1 + \frac{\varepsilon}{\beta \rho}\right)m}, \frac{m}{(1 - \rho + \varepsilon)q^* + \left(1 - \frac{\varepsilon}{\beta \rho}\right)m} \right\}
\]

The following lemma provides a necessary and sufficient condition for ordering the two inequalities.

**Lemma 1 (Thresholds ordering)**

\[
\frac{m}{(1 - \rho - \varepsilon)q^* + \left(1 + \frac{\varepsilon}{\beta \rho}\right)m} \preceq \frac{m}{(1 - \rho + \varepsilon)q^* + \left(1 - \frac{\varepsilon}{\beta \rho}\right)m} \quad \text{if and only if} \quad -u''(x) \frac{x}{u'(x)} \gtrsim 1
\]

**Proof.** In the appendix. \( \blacksquare \)

Note that in this equilibrium open market operations as represented by \( \delta \) do not influence real quantities, so that prices move in steady state one to one with nominal quantities, nor interest rate in the interbank market. Monetary policy can influence real quantities (and so consumption) only through the growth rate of money \( \gamma \). As clear from (27), an increase in \( \gamma \) decreases real money holdings \( m \) lowering consumption for buyers who are in cash only meetings.

2.5.2 **Scarce government bonds equilibrium:** \( \frac{\beta}{\gamma} < \psi < 1 \) and \( \gamma > \beta \)

In this equilibrium, there is scarcity of government bonds. This implies that the price of the bond increases (therefore bonds earn now a liquidity premium) and the borrowing constraint in the interbank market becomes binding. Given the binding borrowing constraint, type 1 bank will trade-off utilities of its depositors according to:

\[
u'(q^{11}) = Ru'(q^{21})
\]

(28)

where the marginal benefit of taking one more unit of cash on the interbank market is given by the marginal utility of the buyers in cash only meetings that will use it, and its marginal cost is given by the interest rate on the interbank \( R \) and the unit more of bonds that must be posted as collateral that therefore decreases consumption of the buyers in meetings with claims.

On the other hand, whenever there is an interbank bank, type 2 bank will equate
consumption of its depositors according to:

$$u'(q^{12}) = Ru'(q^{22})$$  \hfill (29)

where the marginal cost of giving one more unit of cash to the type 1 bank is given by decreasing the amount of money given to depositors in cash only meeting. However since type 1 bank pays an interest $R$ on the interbank, and this is backed by collateral, bank 2 can emit more claims on its deposits, therefore increasing consumption of buyers in not cash only meetings.

Given (28) and (29), the interbank interest rate $R$ has the role equate relative marginal utilities of buyers.\textsuperscript{11}. As also evident from (28) and (29), only one condition of the standard deposit contract needs to be imposed. $R$ will be therefore such that marginal utilities are equalized and given by

$$\frac{1}{\psi} = R = \left(\frac{1}{\delta} - 1\right) \frac{\rho}{1 - \rho} < \frac{\gamma}{\beta}. \hfill (30)$$

Banks still provide complete insurance to all depositors: buyers that use only money consume $q = \frac{m}{\rho}$, where $m$ is still fixed by (27), and the amount of interbank lending is unchanged from the plentiful government bonds equilibrium, $l = \frac{\varepsilon}{\rho} m$. However buyers that use claims on deposits will consume a quantity $q$ such that $\frac{m}{\rho} < q < q^*$, that is they will consume less than the optimal quantity but more than buyers that can use only cash. In this equilibrium open market operations affect the interbank interest rate, the amount and price of bonds and therefore the consumption of buyers that use also bonds in the DM market. For $\delta \uparrow \rho$, we have $R \downarrow 1$, $\psi \uparrow 1$ and $q^{21} = q^{22} = \frac{m}{\rho}$.

The existence of this equilibrium requires $R > 1$, which from (30) implies $\delta < \rho$. The following lemma provides and ordering of the threshold with respect to the previous equilibrium.

**Lemma 2 (Threshold less than $\rho$)** For any $\gamma \geq \beta$, $\delta \leq \rho$, with equality when $\gamma = \beta$.

**Proof.** In the appendix. \hfill ■

Therefore this equilibrium exists for $\gamma > \beta$ and $\delta \leq \delta < \rho$. There are some technical feature of this equilibrium that need to be commented upon. The first is that, as stated before, only one condition of the standard contract needs to be imposed. The second feature of the solution is that, when passing from the equilibrium when $R = \frac{\gamma}{\beta}$ to the equilibrium when $\frac{\gamma}{\beta} > R > 1$ there is a discontinuity of $R$ at $\delta$ (equivalently at $\psi(\delta)$). This is shown in the following lemma.

**Lemma 3 (Discontinuity at $R(\delta)$)** For any $\gamma > \beta$, if $-\frac{u''(x)x}{u'(x)} \geq 1$ then $R(\delta) \leq \frac{\gamma}{\beta}$, with equality when $-\frac{u''(x)x}{u'(x)} = 1$.

\textsuperscript{11}Note that prices equating relative marginal utilities is a standard feature of general equilibrium model.
Proof. In the appendix

The reason for this discontinuity lies in imposing the standard contract condition \( q^{11} = q^{12} \) (or equivalently \( q^{22} = q^{21} \)). When this holds, the loans on the interbank market have the role of equalizing consumption across bank’s types of buyers in cash only meeting. Given this, the interest rate has to adjust so that the interbank market clears. This can create discontinuities at the threshold, since in the plentiful bond equilibrium the standard contract condition did not need to be imposed. In lemma 3 we show that, under a standard hypothesis on the size of the relative risk aversion coefficient, then the interest rate on the interbank market jumps down at \( R(\delta) \).  

2.5.3 Liquidity trap: \( \psi = 1 \) and \( \gamma > \beta \)

When \( \psi = 1 \) money and government bonds are perfect substitutes and therefore, if an interbank market exists, it must be that \( R = 1 \). The nominal interest rate in this case is zero: there is a liquidity trap. In this equilibrium banks have to perfectly equate marginal utilities of both buyers in a cash only meeting and buyers using claims, that is \( u'(q^{1j}) = u'(q^{2j}) \). Buyers consume the same amount of goods 1 independently on the type of banks, without imposing the standard contract condition.

Consumption in all states is less than the optimal quantity, and in particular is given by \( \frac{m}{\delta} \) which satisfies:

\[
\frac{\gamma}{\beta} = u' \left( \frac{m}{\delta} \right)
\]

(31)

Differently than Williamson (2012), our liquidity trap equilibrium is characterized by an indeterminacy about the volume of interbank lending. Since the interbank market is frictionless and there is no cost of borrowing (\( R = 1 \)), for a given monetary stance \( (\gamma, \delta) \) we can have a continuum of levels of interbank lending. In particular, we have two different regions. In one region, where \( \rho \leq \delta < \rho + \varepsilon \), the interbank market is always active and banks exchange at least \( l = \frac{\rho + \varepsilon - \delta}{\delta} m \). In the other region, when \( \delta \geq \rho + \varepsilon \), money is so abundant that is sufficient also for type 1 bank’s needs and an equilibrium in which the interbank market is not active, \( l = 0 \), exists. In the first region, we have a continuum of equilibria Pareto equivalent in which the amount of interbank lending \( l \) is in the interval \( \left[ \frac{\rho + \varepsilon - \delta}{\delta} m, \frac{\varepsilon}{\rho} m \right] \). In the second region \( l \) is in the interval \( \left[ 0, \frac{\varepsilon}{\rho} m \right] \). It should be noted, however, that the implication of this result is that there exist equilibria in which banks of type 1 borrows from banks of type 2 an amount of money greater than their need and keep this money as reserves in the CM. All these equilibria exist because the interbank market is frictionless, but, obviously, introducing a small transaction cost the only robust equilibrium is the one in which banks of type 1 borrow what they need.

\[\text{Under the opposite assumption on the relative risk aversion coefficient (less than 1), then we can show that } R(\delta) \text{ is higher than } \frac{\gamma}{\beta}, \text{ however by proposition 2 this cannot be an equilibrium. Therefore under the assumption } -\frac{u''(x)}{u'(x)} < 1 \text{ there is a region of } \delta \text{ where an equilibrium with standard contract cannot exist.}\]
Monetary policy choice $\delta$ has no real effects in the liquidity trap equilibrium, however real money holdings are not constant anymore. An increase in $\delta$ increases proportionally real money holding $m$, that is, the relative price of goods with respect to money does not change in a liquidity trap. However, monetary policy does influence activity on the interbank market, since $l$, the amount exchanged, is decreasing in $\delta$.

Figure 3 and 4 summarize the effects of a change in $\delta$ on interbank market price $R$ and quantity exchanged $l$ and on buyers’ consumption when $\gamma > \beta$.

2.5.4 Equilibrium at the Friedman rule

This framework always allows for a Friedman rule equilibrium, i.e. when $\gamma = \beta$. Then from (25) $\lambda_1 = \lambda_2 = 0$ and $q^{11} = q^{12} = q^{21} = q^{22} = q^*$. Moreover, from (26) the price of the government bond is $\psi = 1$. Therefore, when the central bank follows the Friedman rule she gets optimal consumption independently on the value taken by $\delta$.

The intuition behind this result is that at the Friedman rule there is no cost associated to holding fiat money and therefore banks are disposed to hold any amount of it. Here, as
Figure 4: Effect of monetary policy parameter \( \delta \) on consumption when \( \gamma > \beta \). Solid line: consumption of buyers in claims meetings. Dotted line: consumption of buyers in cash only meetings. \( q^* : u'(q^*) = 1 \), \( \tilde{q} : u'(\tilde{q}) = \gamma/\beta \)

we saw from (26), the price of the bond is \( \psi \equiv 1 \), but differently from the liquidity trap equilibrium there is no liquidity premium that keep high the price of the bond.

Moreover, at the Friedman rule money and government bonds are equivalent in their fundamentals, therefore the liquidity insurance role of banks disappears and the existence of a banking system is no more Pareto improving.

3 Model with Money, Bonds and a Short Lived Real Asset

We extend the baseline model introducing into the economy the possibility to trade claims on a short-lived Lucas tree. In any period \( t \) during the CM, a real asset is created in fixed net supply \( A > 0 \) and claims on it are given in equal proportion to buyers.\(^{13}\) Agents (and banks) can trade the asset at price \( p \) in the CM. The asset pays off at the beginning of the CM in period \( t + 1 \) an uncertain dividend (in units of good 2) \( k \) which, for simplicity, can take values \( k_h \) and \( k_l \), \( (k_h > k_l) \) with probability \( 1 > \pi > 0 \) and \( 1 - \pi \). Let \( \bar{k} \) indicate the unconditional average pay off of the asset, \( \bar{k} \equiv \pi k_h + (1 - \pi) k_l \). The realization of the dividend is not known by anybody before the beginning of the CM in period \( t + 1 \).

In a fraction \( \rho \) of meetings in the DM claims on the real asset cannot be used as medium-of-exchange. However, as before banks can use the real asset as guarantee for claims on deposits that can be traded in a fraction \( 1 - \rho \) of exchanges in the DM. Also, similar to the role of government bonds in the model, banks can use the asset as collateral in the interbank market.

Crucially, the differences in the intrinsic characteristics of government bond and real asset make the former less liquid than the latter. While bonds are risk-free, real assets have

\(^{13}\)The linearity of preferences during the CM implies that is anyway not important to whom, and in which proportion, we give the asset.
stochastic returns. The realization of the dividend of the real asset is public knowledge at the beginning of the CM before interbank loans settlement. Then, in a low-dividend state type 1 banks have the incentive to declare default on interbank loans, letting type 2 banks seize the collateral. Anticipating that, the real asset cannot be pledged in the interbank market in the DM for a value higher than its lowest dividend $k_l$, notwithstanding his expected value is $\bar{k} > k_l$.

Differences in liquidity properties emerge also when banks use the real asset as guarantee for claims on deposits. Since sellers cash these claims in the CM, they also face the risk to get not repaid when the real asset is pledged for a value higher than $k_l$. Therefore, whenever there is uncertainty about returns and an incentive to default, the real asset, differently from bonds, is worth only a fraction of his expected value, $k_l < \bar{k}$.14

The aim of our setup is to introduce over-collateralization of the real asset, i.e. it is subject to a positive haircut when used as collateral15. Given the linearity of the utility function in the CM and the uncertainty about the realization of the dividend, in the DM one unit of the real asset is worth to agents and banks $\bar{k}$. If the asset pays off at the end of the CM, at the moment of the settlement of loans the asset is still worth to agents and banks $\bar{k}$, because there is uncertainty about the return. Therefore, in the DM the real asset can be pledged at a value $\bar{k}$, because at the settlement the asset will still be still worth $\bar{k}$ and there is no incentive for the debtor to declare default. In this case the real asset is not subject to an haircut and it is fully pledgeable as the government bonds. However, as we explained above when the settlement takes place after the realization of the dividend there is the incentive for the debtor to declare default in a low-dividend state of the world if the asset is pledged at a value greater than $k_l$. Therefore, the distance between $\bar{k}$ and $k_l$ is a measure of the haircut applied in the DM to the real asset. It should be noted that what give rise to over-collateralization is not the riskness of the asset, but the key factors are: i) the absence of a commitment technology; ii) the potential decline of the value of the asset between the moment the credit contract is drawn and the settlement.

We anticipate that in our framework a mean preserving spread of the distribution of the dividends (a greater volatility) increases the haircut on the real asset, because notwithstanding the expect value in the DM is unchanged the return in the low-dividend

---

14This result can also be easily derived by requiring that banks cannot default. Suppose a bank has asset $a$ and pledged an amount $\hat{a}^1 \leq a$ of its assets at a value $k > k_l$, that is it pledged $k\hat{a}^1$. In the CM the bank will have to pay off the claims on its asset emitted during the DM in any state of the world. This implies that, if the bank cannot default, we will have in the low state that $ak_l - k\hat{a}^1 \geq 0$. When the bank is pledging all its asset, $\hat{a}^1 = a$, then it must be that $k \leq k_l$. Since it is not welfare maximizing to pledge at a value strictly less than $k_l$, then $k = k_l$.

15In the real world, when a lender makes a collateralized loan to a borrower the accorded loan is worth only a fraction of the current value of the asset pledged as collateral, in order to reduce the credit risk of the lender in an event of default. The difference between the current value of collateral and the loan accorded is the haircut applied to the collateral. Since the value of the asset may decline in the period between the credit contract is drawn and the settlement, the presence of the haircut allows the lender in an event of default to recover the loan selling the collateral at his current value.
state is now lower. In the following we show how this can have strong effects on the equilibrium prices and quantities and, therefore, interesting policy implications.

3.1 Centralized Market

Differently from the previous case, the banks will now decide also how much asset to carry in the DM, that is they solve:

$$
\max_{m,b,a} -\gamma m - \psi \gamma b - pa + \beta EV(m,b,a)
$$

giving rise to the following first order conditions:

$$
\frac{\gamma}{\beta} = EV_m(m,b,a) \quad (32)
$$

$$
\frac{\psi \gamma}{\beta} = EV_b(m,b,a) \quad (33)
$$

$$
\frac{p}{\beta} = EV_a(m,b,a) \quad (34)
$$

3.2 Decentralized Market

In the DM there will as before two types of banks, depending on the liquidity shock they face. The bank that has higher than average depositors in need of money, a type 1 bank, will solve the following problem:

$$
\max_{\hat{m}^1,\hat{b}^1,\hat{a}^1,l} (\rho + \varepsilon)u\left(\frac{\hat{m}^1 + l}{\rho + \varepsilon}\right) + (1 - \rho - \varepsilon)u\left(\frac{b - \hat{b}^1 + k_l(a - \hat{a}^1) + m - \hat{m}^1}{1 - \rho - \varepsilon}\right) + \\
\tilde{k}(a - \hat{a}^1) - k_l(a - \hat{a}^1) + (\hat{b}^1 + \tilde{k}\hat{a}^1 - Rl) + k_l(a - \hat{a}^1)
$$

s.t. \ \hat{a}^1 \geq 0, \ \hat{b}^1 \geq 0, \ l \geq 0

a \geq \hat{a}^1, \ b \geq \hat{b}^1, \ m \geq \hat{m}^1

Rl \leq \hat{b}^1 + k_l\hat{a}^1

where the only difference with the model without asset is that now the bank can pledge both bonds and asset in the interbank market. The bank with less than average depositors

---

\footnote{Since both buyers and sellers have linear utility functions in the CM, they are risk neutral agents. Then, in a pure walrasian economy the volatility of the returns does not matter, because agents care only about the expected return. Here, instead, volatility matters because affects the degree of pledgeability of the real asset.}
who needs money, bank 2, will instead solve the following problem:

$$\max_{\hat{m}^2, \hat{b}^2, \hat{a}^2, l} (\rho - \varepsilon)u\left(\frac{\hat{m}^2}{\rho - \varepsilon}\right) + (1 - \rho - \varepsilon)u\left(\frac{b - \hat{b}^2 + k_l(a - \hat{a}^2) + m - \hat{m}^2 - d + Rd}{1 - \rho + \varepsilon}\right) +$$

$$+ \bar{k}(a - \hat{a}^2) - k_l(a - \hat{a}^2) + (\hat{b}^2 + \bar{k}\hat{a}^2)$$

s.t. \(\hat{a}^2 \geq 0, \hat{b}^2 \geq 0, d \geq 0\)

\(a \geq \hat{a}^2, b \geq \hat{b}^2, m \geq \hat{m}^2 + d\)

### 3.3 Equilibria of model with money, bonds and a short lived asset

The definition of equilibrium in the centralized and decentralized market is similar to before, the only addition that also the asset market needs to clear, therefore \(a = A\). The first order conditions of the problems are specified in the appendix. Here, we start providing some feature of the equilibria:

**Lemma 4 (Indeterminacy of \(\hat{a}^1\) and \(\hat{b}^1\))** Type 1 bank will always use either both real assets and bonds or neither of them in the interbank market. When both are used and \(\hat{b}^1 < b\) and \(\hat{a}^1 < a\), then the bank is indifferent between using bonds or real asset.

**Proof.** In the appendix

The intuition for the previous lemma is straightforward. Bonds and real assets give rise to the same trade-off between the marginal cost of reducing deposits available for buyers’ consumption and marginal benefit of posting one unit more of either of them on the interbank market as collateral. This is so since one unit of pledged asset increases the collateral pool by \(k_l\) and reduces deposits by \(k_l\), and one unit of pledged bond increases the collateral pool by one and reduces deposits by one, or said differently, they have the same opportunity-cost. A corollary of the lemma is that it cannot be the case that the bank pledges all its bonds (real assets) in the interbank market but holds some positive amount of real assets (bonds) as excess reserves. Note that the previous lemma is true also for bank 2, even though the reason is slightly different: real assets and bonds have the same marginal effect on utility when used in the DM or when kept as excess reserves.

The previous lemma tells us that we cannot pin down exactly \(b\) and \(a\) (unless both of them are zero or \(b\) and \(a\) respectively), therefore we introduce in our analysis a new variable, \(\pi\), that represents the total value of interest bearing assets in the portfolio. Specifically we let:

$$\pi \equiv b + k_l a, \quad \hat{\pi} \equiv \hat{b}^i + k_l \hat{a}^i, \quad i = 1, 2. \quad (35)$$

$$\hat{\pi}^i \equiv \hat{b}^i + k_l \hat{a}^i, \quad i = 1, 2. \quad (36)$$
For the sake of convenience in the analysis, we will again define the following quantities:

\[ q^{11} \equiv \frac{\hat{m}^1 + l}{\rho + \varepsilon}; \quad q^{21} \equiv \frac{\pi - \hat{\pi}^1 + m - \hat{m}^1}{1 - \rho - \varepsilon}; \]

\[ q^{12} \equiv \frac{\hat{m}^2}{\rho - \varepsilon}; \quad q^{22} \equiv \frac{\pi - \hat{\pi}^2 + m - \hat{m}^2 - d + Rd}{1 - \rho + \varepsilon}. \]

(37)

The following proposition provides some characterization of the prices in equilibrium.

**Proposition 3 (Equilibrium prices)** In every equilibrium \(\beta \gamma \leq \psi \leq 1\) and \(\beta \bar{k} \leq p \leq \beta \bar{k} + \beta k_1 (\frac{\gamma}{\bar{k}} - 1)\). Moreover, whenever the interbank market exists \(R = \frac{1}{\psi}\).

**Proof.** In the appendix  ■

The price of bonds and of liquidity on the interbank market are the same as in the model without the asset. The price of the asset in equilibrium has to be at least equal to \(\beta \bar{k}\). This is the standard pricing equation that would arise in steady state in any general equilibrium asset pricing model. The price of the assets reflects its expected dividend discounted by \(\beta\) (Note that in steady state the ratio of marginal utilities of today’s and tomorrow’s consumption are equal to each other). Here, the price of the asset can also have a liquidity premium, that is a premium commanded by the asset given by the technological feasibility of being used in the exchange process. When in the following section we analyze the different equilibria of the model it will be clear in which instances the asset offers its liquidity premium.

### 3.3.1 Plentiful interest bearing assets equilibrium

As in section 2.5.1, in this equilibrium the amount of interest bearing assets is plentiful. Banks have in their portfolio enough assets such that they can provide to their depositors in a claim meeting enough liquidity such that they can consume the optimal amount of good 1, \(q^*\). At the same time, the borrowing constraint for type 1 banks in the interbank market is not binding, \(Rl < \hat{\pi}^1\).

In this case, at the margin neither bonds nor real assets relaxe the borrowing constraint or allows banks to make their depositors consume more. Therefore, there is no liquidity premium on bonds and the gross real interest rate on nominal bonds is \(\frac{\phi_{t+1}}{\phi_t} \frac{1}{\psi} = \frac{1}{\gamma \beta} = \frac{1}{\beta}\), that is, it equals the rate of time preference. The price of the real asset is always related to bond price by the following no arbitrage condition

\[ p = \beta \bar{k} + k_1 (\psi \gamma - \beta) \]

(38)

Therefore, in this equilibrium the price of the real asset is simply \(p = \beta \bar{k}\), i.e. the price reflects only the fundamentals.

In this equilibrium since prices of bonds and real assets are so low, banks are indifferent to carry any amount of interest bearing assets needed to issue claims to finance buyers.
and, moreover, they will carry excess reserves to the next centralized market $\pi^1 > 0$ and $\pi^2 > 0$.

As in the case in section 2.5.1, type 1 banks will go on the interbank market to obtain more cash for its depositors, taking a loan $l = \frac{\varepsilon}{\rho} m$ and paying an interest rate $R = \frac{\gamma}{\beta} > 1$. Since the constraint on the interbank market is not binding, the marginal cost and benefit of having one more unity of money for both banks are equal, therefore also consumptions of depositors in cash only meeting across the two banks will be equalized without imposing the standard contract. Therefore $q^{11} = q^{12} = \frac{m}{\rho} < q^* m$ is fixed by (27).

For this equilibrium to exists $\delta$ must be low enough such that in the market there is plentiful of bonds, or that there is an high amount of real assets. Indeed, there is a threshold value for amount of real assets, $\bar{A}(\gamma)$, such that for $A \geq \bar{A}(\gamma)$ for any $\delta \in (0,1]$ this is the unique equilibrium.

**Proposition 4** Define $\bar{A}(\gamma)$ as

$$
\bar{A}(\gamma) \equiv \min \left\{ \frac{(1 - \rho - \varepsilon)q^* + \frac{\gamma}{\beta} \frac{\varepsilon}{\rho} m}{k_l}, \frac{(1 - \rho + \varepsilon)q^* - \frac{\gamma}{\beta} \frac{\varepsilon}{\rho} m}{k_l} \right\}
$$

For a given $\gamma$, if $A \geq \bar{A}(\gamma)$ there exists only one equilibrium with $\psi = \frac{\beta}{\gamma}$ and $p = \beta \bar{k}$.

**Proof.** Since we are looking for the minimum amount of real asset that allow to consume $q^{21} = q^{22} = q^*$, we consider the case in which $B_t = 0 \forall t$ or, equivalently, $\delta = 1$. Rearranging the two disequalities, using $\delta = 1$, we can easily find the expression for $\bar{A}$. For $A \geq \bar{A}$, changing $\delta$ in the interval $(0,1]$ does not change any quantity or price. ■

When $A \geq \bar{A}(\gamma)$, the central bank can affect the equilibrium only changing $\gamma$ because the value taken by $\delta$ is irrelevant. However, when $A < \bar{A}(\gamma)$ there exists a threshold value $\delta^1$ such that the existence of this equilibrium requires $\delta < \delta^1$ (see in the appendix).

When $A < \bar{A}(\gamma)$ a marginal change of $\delta$ does not influence real quantities. In fact, notwithstanding the lower amount of bonds there are still enough interest bearing assets such that consumption is optimal and the borrowing constraint is slack. Moreover, from (27) the real amount of money is determined only by $\gamma$. Therefore, an injection of fiat money would result in a proportional increase in the price level, without affecting the consumption of cash-only buyers. As before, only a change in $\gamma$ have real effects on consumption.

### 3.3.2 Equilibrium with scarce interest bearing assets

When interest bearing assets are scarce, banks cannot give to their depositor in claim meetings enough claims such that they can buy the optimal quantity of good 1. Moreover, borrowing constraint for type 1 banks is binding. In this situation, at the margin one, government bonds (or real assets) relaxe the borrowing constraints and allows banks to give
greater consumption to their depositors. Therefore, interest bearing assets are valued not only for their fundamentals, but also because they facilitate exchange. This implies that the prices of bonds and real assets increase (therefore interest bearing assets earn now a liquidity premium) and the borrowing constraint in the interbank market is binding. However, there are enough assets such that the price of bonds is less than 1.

As before, given the binding borrowing constraint type 1 bank will trade-off utilities of its depositors according to (28) and type 2 banks according to (29). Therefore, the interbank interest rate $R$ has the role to equate relative marginal utilities of buyers. As also evident from (28) and (29), only one condition of the standard deposit contract needs to be imposed. Also in this case $R$ will such be lower than $\frac{\gamma}{\beta}$ and decreasing in $\delta$.

Banks provide incomplete insurance to depositors: buyers that use only money consume $q = \frac{m}{p}$, where $m$ is still fixed by (27), and the amount of interbank lending is unchanged from the plentiful government bonds equilibrium, $l = \frac{\varepsilon}{p}m$. However buyers that use claims on deposits will consume a quantity $q$ such that $\frac{m}{p} < q < q^*$, that is they will consume less than the optimal quantity but more than buyers that can use only cash. In this equilibrium open market operations affect the interbank interest rate, the bond market and therefore the consumption of buyers that use also bonds in the DM market. For $\delta > \rho$, we have $R \downarrow 1$, $\psi \uparrow 1$ and $q^{21} = q^{22} \downarrow \frac{m}{p}$.

The existence of this equilibrium requires $R > 1$, which implies $\delta < \bar{\delta}$, where $\bar{\delta}$ is defined in the following section.

### 3.3.3 Equilibrium with liquidity trap

When the amount of interest bearing assets is low, banks cannot give to their depositors in claim meetings enough claims such that they can buy the optimal quantity of goods. Moreover, borrowing constraint for type 1 banks is binding. In this situation, at the margin, government bonds (or real assets) relax the borrowing constraints and allow banks to give greater consumption to their depositors. Therefore, interest bearing assets are valued not only for their fundamentals, but also because they facilitate exchange. When the amount is excessively low, prices increase until $\psi = 1$ and $p = \beta \bar{k} + k_l(\gamma - \beta)$.

When $\psi = 1$ money and government bonds are perfect substitutes therefore, if an interbank market exists, it must be that $R = 1$. The nominal interest rate in this case is zero: there is a liquidity trap. In this equilibrium banks have to perfectly equate marginal utilities of both buyers in a cash only meeting and buyers using claims, that is $u'(q^{1j}) = u'(q^{2j})$.

As in the case with only money and bonds, if the amount of real asset is sufficiently low there exists an equilibrium in which there is no interbank market, because banks have a sufficient amount of fiat money in every state. When the amount of the real asset is too high, this equilibrium never exists for $\delta$ in the interval $(0, 1]$. The following proposition define the value $A$ such that for $A \geq A$, this equilibrium never exists.
Proposition 5 Define $A$ such that $m > 0$ and the following conditions hold:

$$
\frac{\gamma}{\beta} = u'(m + k_l A)
$$

$$
1 = \frac{1}{\rho + \varepsilon} - \frac{k_l A}{m}
$$

If $A \leq A$ an equilibrium with liquidity trap and no interbank market exists for a $\delta^2$ sufficiently high in the interval $(\rho + \varepsilon, 1)$.

Proof. In the appendix. ■

When $A < A$, there always exists four different equilibrium regions as in the case without real assets. When $A = 0$, the equilibrium without interbank markets exists for $\delta \geq \rho + \varepsilon$, while in the case with $0 < A < A$ we need $\delta \geq \delta^2 > \rho + \varepsilon$.

As before, if $\delta^1 < \delta < \delta^2$, where $\delta^1 > \rho$, the interbank market is active and banks exchange $l < \frac{\varepsilon}{\rho} m$. Also here the amount of interbank lending is decreasing in $\delta$ and it is equal to zero at $\delta = \delta^2$. Consumption in all states is less than the optimal quantity, and in particular is given by $\frac{m}{\delta} + k_l A$ which satisfies:

$$
\frac{\gamma}{\beta} = u'\left(\frac{m}{\delta} + k_l A\right)
$$

(39)

In the other region, when $\delta \geq \rho + \varepsilon$, money is so abundant that is sufficient also for type 1 bank’s needs, therefore the interbank market is not needed anymore to provide insurance. Here $l = 0$ but consumption is always defined by (31).

Monetary policy choice $\delta$ has no real effect in the liquidity trap, however real money holdings are not constant anymore. An increase in $\delta$ increases proportionally real money holding $m$, that is, the relative price of goods with respect to money does not change in a liquidity trap. However monetary policy does influence activity on the interbank market, since $l$, the amount exchanged, is decreasing in $\delta$.

3.3.4 Summary of Equilibria

From the previous analysis of equilibria, when a real asset is present in the economy, the total amount of real asset in the economy $A$ affects the threshold between the different types of equilibria and it also determines which types of equilibria can be found in the economy. Figure 5 shows the types of equilibria in the economy as function of $A$ and the monetary policy parameter $\delta$.

As discussed before, when $A < A$, then all four types of equilibria are found in the economy. However as $A$ increases the types of equilibria decreases, until, when $A > \bar{A}(\gamma)$, only the plentiful interest bearing asset equilibrium survives. Under the assumption $A < A$, figure 6 summarizes the effect of a change in $\delta$ on interbank market price $R$ and quantity exchanged $l$. 

27
Figure 5: Equilibria of the model with respect to $A$ and $\delta$ (LT stands for liquidity trap). Thresholds $\bar{A}$, $\tilde{A}$ and $\hat{A}$ are defined in the appendix.

Figure 6: Effect of monetary policy parameter $\delta$ on interbank market when $\gamma > \beta$ and $A < \bar{A}$ in model with real asset.
Introducing the real asset in the economy increases all thresholds to the right. The intuition for this is that, since the pledgeability of the real asset increases the collateral pool of the economy, a higher fraction of money with respect to nominal bonds is needed in order for interest bearing asset to become scarce. This implies that the probability for the economy of falling into a liquidity trap decreases (a smaller area of the parameter space implies a liquidity trap equilibrium).

Moreover, not only thresholds increase, but also the absolute distances between them increase. Therefore, with respect to the model without the real asset, a greater open market operations (a greater increase in $\delta$) is needed in order to have the same effects. The following proposition shows formally this feature of the equilibria in the economy:

Proposition 6 $\bar{\delta}_1^1, \bar{\delta}_1^1$ and $\bar{\delta}_2^1$ are strictly increasing in $A$. Moreover $\bar{\delta}_1^1 - \delta_1^1$ and $\bar{\delta}_2^1 - \delta_1^1$ are strictly increasing in $A$.

Proof. In the appendix.

3.4 Open Market Operation: swap of real assets for government bonds

The presence of a real asset does not alter the effect of a standard open market operation, a swap of fiat money for government bonds, with respect to the model in which the real asset is not present as analyzed before in section 2.5, modulo the larger operation size needed to have the same effect. However, during the financial crisis, Central Banks of developed economies performed unconventional monetary policies that involved the purchase of risky assets. Does it matter for the equilibria in the economy whether the purchases of risky assets were financed through reserves or risk-free bonds? This section and the following are aimed at answering this question in the context of our model. Here we discuss an open market operation that swaps real assets for government bonds, while in the next section we analyze an open market operation that swaps real assets for currency.

While so far open market operations were represented by a change in $\delta$, we describe the purchase of real assets as a one-time increase in government liabilities (money and government bonds) and a reduction of the amount of real asset in the hands of the private sector. This implies that $\delta$ is no longer an exogenous parameter, but it changes depending on the nominal quantity of money or government bonds needed to perform the operation.

Let us consider the effects of an open market purchase of real assets issuing new bonds. This policy is equivalent to a contemporaneous reduction of $A$, the amount of real assets in the private sector, and a reduction of $\delta$, because financing the purchase with additional bonds reduces the ratio of money on total nominal liabilities. However, what matters for determination the equilibrium is the relative scarcity of interest bearing assets with respect to money in the decentralized market. A reduction of $\delta$ implies that the amount
of government bonds increases, but there is also a contemporaneous decrease of $A$, the amount of risky real asset in the economy. Therefore, in order to analyze the effect of the operation on equilibrium quantities and prices, we need to determine its net effect on the amount of pledgeable interest bearing assets.

We suppose that the consolidated government keeps the commitment to let $M_t + B_t$ increase at a rate $\gamma$, but at time $t$, during the centralized market, it buys an amount $A_0 < A$ of real assets issuing an extra-amount $B^0$ of new nominal bonds.\(^{17}\) At time $t$ the nominal amount of bond $B^0$ issued must be given by

$$\phi B^0 \psi = pA^0 = \left[ \beta \bar{k} + kl(\psi \gamma - \beta) \right] A^0$$

since the real amount of resources that the government levies issuing bonds, $\phi B^0 \psi$, has to be equal to the amount of resources needed in order to buy in the centralized market an amount $A_0$ of real assets, that it is equivalent to $pA^0$. Note that $\phi$ does not change with the operation since the nominal amount of money did not change. The nominal amount of bond issued is therefore equal to:

$$B^0 = \frac{1}{\phi \psi} \left[ \beta \bar{k} + kl(\psi \gamma - \beta) \right] A^0.$$ 

Suppose we are not in the liquidity trap equilibrium, then the real value of nominal liabilities is always pinned down by the amount of fiat money\(^{18}\)

$$\frac{\gamma}{\beta} = u' \left( \frac{\phi' M_t}{\rho} \right)$$

where $\phi'$ is the price of money in terms of the real good during the subsequent (with respect to the time of the operation) decentralized market. Since the consolidated government keeps the amount of fiat money growing at rate $\gamma$, we have $\frac{\phi}{\phi'} = \gamma$. The real value of $B^0$ in the subsequent decentralized market is equal to

$$b^0 \equiv \phi' B^0 = \frac{\phi'}{\phi \psi} \left[ \beta \bar{k} + kl(\psi \gamma - \beta) \right] A^0 = \left[ \frac{\beta}{\gamma \psi} + kl \left( 1 - \frac{\beta}{\gamma \psi} \right) \right] A^0.$$ 

Therefore after the open market operation swapping the real asset for government bonds,\(^{19}\)

\(^{17}\)Since the real asset is one-period-lived, the operation should be rolled over every period, but the change in the total amount of real government liabilities is only at time $t$. Note that $\delta$ adjust endogenously, therefore influencing the price of bonds and of the real asset. This change will however create only quantitative effect of final net pledgeability, and not qualitative effects.

\(^{18}\)Outside the liquidity trap equilibrium, the real amount of money is determined by (27). This implies that a swap of real assets for government bonds does not affect the real value of nominal liabilities, $\phi_{t+1}$, that it is determined only by the amount of fiat money. Since we assume that the central bank let $M$ increase at a rate $\gamma$, we still have $\frac{\phi}{\phi'} = \gamma$. In a liquidity trap, instead, the real value of nominal liabilities is fixed by (39). In this case swapping real assets for government bonds can affect also $\phi_{t+1}$ and it is not straightforward to analyze the effects on the equilibrium quantities and prices.

\(^{19}\)Note that we are also implicitly assuming that the adjustment to the new steady state is immediate.
the amount of pledgeable interest bearing assets for banks in the decentralized market is given by

\[ \tilde{\pi} \equiv b + b^0 + k_l (A - A^0) = b + k_l A + b^0 - k_l A^0 \]

\[ \tilde{\pi} = \pi + \frac{\beta}{\gamma \psi} (\bar{k} - k_l) A^0. \]

The new amount of pledgeable assets, \( \tilde{\pi} \), is therefore greater than the previous, \( \pi = b + k_l A \), as long as the real asset is partially pledgeable. If the asset offers a certain dividend in every period, \( k = \bar{k} \), or if there are no frictions that induce over-collateralization and banks can pledge \( \bar{k} \), open market operations that swap real assets with bonds are irrelevant, they do not alter the equilibrium of the economy. However when the real asset is only partially pledgeable, the substitution of real assets with bonds increases the amount of pledgeable liquidity for banks. Intuitively the central bank is increasing the quality of the collateral pool in the economy, by taking out a risky asset and increasing the availability of risk-less, fully pledgeable, government bonds. In equilibrium, since the amount of pledgeable asset increases, the price of the assets decreases and the interest rate on the interbank market increases, because interest bearing liquidity is less scarce than before with respect to fiat money. Since the real amount of money is unchanged, also amount of interbank lending is invariant.

Finally, an additional effect we want to highlight is that all the thresholds of the equilibrium regions decreases.

### 3.5 Open Market Operation: swap of real assets with fiat money

While a swap of real assets for government bonds increases the amount of pledgeable interest bearing assets, a purchase of real assets financed with fiat money has the opposite effect. Following the same approach as before, we suppose that we are outside a liquidity trap equilibrium, \( \frac{\gamma}{\beta} \geq R > 1 \). The consolidated government keeps the commitment to let \( M_t + B_t \) increasing at a rate \( \gamma \), but at time \( t \) starts to buy an amount \( A^0 < A \) of real assets printing an extra-amount \( M^0 \) of money\(^{20}\), and we let \( \delta \) adjust endogenously. We suppose also that the amount \( A^0 \) is small, such that after this operation we still are in an equilibrium with \( R > 1 \). In this equilibrium the real value of nominal liabilities in the following decentralized market, \( \phi' \), is pinned down by the condition

\[ \frac{\gamma}{\beta} = u' \left[ \frac{\phi' (M_t + M^0)}{\rho} \right] \]

\(^{20}\)Since the real asset is one-period-lived, the operation should be rolled over every period, but the change in the total amount of real government liabilities is only at time \( t \).
where \( M^0 \) is the extra amount of fiat money injected in order to buy an amount \( A^0 \) of real assets. Since in this equilibrium banks do not keep fiat money as reserves, \( \dot{m}^1 = m \) and \( \dot{m}^2 + l = m \), from the above condition the injection of fiat money \( M^0 \) has the only effect to lower the real value of nominal liabilities, \( \phi' \). The open market operation affects the amount of pledgeable assets through two channels. First the amount of real assets is now \( A - A^0 \) and it is not replaced by fiat money, because it is not held as reserves. Moreover, the injection of fiat money decreases the real value of outstanding bonds, because we assumed that the stock of bonds grows at a rate \( \gamma \). Therefore, this operation reduces the amount of interest bearing assets, lowering \( R \) and increasing prices of the assets, because the liquidity premium increases. Since the real amount of money is unchanged, also amount of interbank lending is invariant. Moreover, the consumption of the buyers in claims meetings decreases. All the thresholds decrease, pushing the economy near the liquidity trap equilibrium.

In a liquidity trap equilibrium, instead, the real amount of money is the solution to \( (39) \) and \( \delta > \bar{\delta}^1 \), as shown before. When the central bank acquires the real assets with fiat money, it is equivalent to a reduction of \( A \) (therefore \( \bar{\delta}^1 \) and \( \bar{\delta}^2 \) shift to the left) and an increase in \( \delta \). This means from \( (39) \) that the real amount of money increases, but the distance from an equilibrium with scarce interest bearing assets increases. This operation can affect at most the amount of interbank lending, that decreases.

It should be noted that in a liquidity trap equilibrium also a swap of real assets for bonds can be ineffective. Substituting the real assets with government bonds keeping unchanged the growth path of fiat money corresponds to a decrease of \( \delta \). However, with a lower amount of \( A \) also the thresholds \( \bar{\delta}^1 \) and \( \bar{\delta}^2 \) shift to the left and, therefore, the equilibrium can be still a liquidity trap.

3.6 Volatility shocks

Notwithstanding agents have linear utility in the CM and therefore they are risk neutral, we want to show how an increase of the volatility of the real asset’s returns affects the equilibrium of the model. In our simple two states framework we can map an increase in volatility as a mean preserving spread, that is, an increase in the difference between the low and high state return of the real asset (with suitable changes in the probability such that the unconditional average is unchanged).

Since banks can pledge the real asset at a value at most equal to \( k_l \), the amount of pledgeable wealth of banks reduces. This implies that: i) banks have less liquidity to offer to their depositors; ii) banks have less collateral to pledge with other banks on the interbank market. If before the increase of volatility the equilibrium was with scarce interest bearing assets, the amount of interbank loans remains the same. However, asset’s liquidity is now more scarce with respect of fiat money and, therefore, the relative price between money and interest bearing assets has to change. In equilibrium the interest rate
on the interbank market has to decrease. Moreover, since $q^{21} = q^{22} = q$ decreases, the prices of the assets, $\psi$ and $p$, increase because the liquidity premium $[u'(q) - 1]$ increases.

A second effect we want to highlight is that the equilibrium thresholds depend on the amount of pledgeable real asset, $k_lA$. Consider the case in which $k_l$ decreases. Then all the thresholds defining the different equilibria shift to the left, that means $\delta^1$, $\bar{\delta}^1$ and $\bar{\delta}^2$ all decrease\(^{21}\). This implies that for a given stance of monetary policy $\delta$, an increase in volatility can shift the economy from its equilibrium. In particular if the economy is the scarce interest bearing asset equilibrium, then an increase in volatility, by reducing the quality of available collateral, pushes the economy towards the liquidity trap equilibrium. The effect of an increase in volatility is shown in figure 7.

\[ R \]

Red line: $k_l > 0$
Blue line: $k_l = 0$

Figure 7: Effects of a mean spread change on asset’s return

4 A glance at the data

Our model has several qualitative implications regarding rates on the interbank market. In particular it implies: i) rates are constant either for very low or for very high liquidity provision of the Central Bank; ii) rates decrease with liquidity for intermediate liquidity provision of the Central Bank, where intermediate liquidity thresholds change with volatility in the economy. In order to see whether predictions match the data we provide here some evidence on liquidity provided by the ECB and interest rates on the interbank market.

The interbank market, since it is mostly composed of over the counter transaction, it unfortunately provides paucity of data. Moreover the large different types of collateral and the different goal of repo operations\(^{22}\) imply that a single rate figure will be, at best,

\(^{21}\)Clearly, in the limit case in which $k_l = 0$ (or $A = 0$), the thresholds are not different from the model with only government bonds.

\(^{22}\)Banks can access the interbank market either for obtaining liquidity, but also for obtaining specific type of collateral needed for other operations.
an incomplete match with our theoretical counterpart.

Here we focus on European data, and in particular, as a reference rate of the collateralized interbank market, we rely on the Eurepo® rate on tomorrow/next transactions. The Eurepo® rate is based on daily survey of major banks in Europe by the European Banking Federation, and is defined as “the rate at which, at 11 a.m. Brussels time, one bank offers, in the euro-zone and worldwide, funds in euro to another bank if in exchange the former receives from the latter the best collateral withing the most actively traded European repo market.” In figure 8 we provide a scatter plot of daily values of Eurepo rate (minus the rate of the Deposit facility at the ECB, in order to correct for changes in the interest rate over the sample period) against liquidity provided by the ECB through their refinancing operations over the sample period 1/1/2004 to 15/1/2013. Given that the ECB performs refinancing operations through repurchase agreements with collateral defined by ECB, this measure stands as a direct empirical approximation of the variable δ in the model.

The first observation when looking at the figure is that there are two ranges of the values for the liquidity over which our measure of the interbank rate is fairly constant. This happens for low values of liquidity (less than around 450 billions of euro) and for large values of liquidity (greater than around 900 billions of euro). As in the model, and not surprisingly, the interest rate charged on the interbank market is higher when liquidity is scarce than when liquidity is abundant.

The second, and more interesting, observation is that there exists a range of values of liquidity provided over which the rate on the interbank market appears to be monotonically decreasing in liquidity (roughly between 450 and 900 billions of euro). This is similar to how our model predicts the rate on the interbank market moves in the scare interest bearing asset equilibrium.

The long sample range implies that observations come from periods that were very

---

23There 26 banks in the panel, a smaller sample than the more widely used Euribor indicator, which provides rates on uncollateralized money market, and collected by the same sponsor. The list of panel banks can be accessed at www.euribor-ebf.eu/eurepo-org/panel-banks.html and historical data is available on the same website.

24Eurepo official documentation: www.euribor-ebf.eu/eurepo-org/about-eurepo.html. The survey does not indicate which is the type of collateral to which the rate applies in each day.

25Note that this difference effectively measure the interest rate on providing liquidity on the interbank market, since banks can always deposit excess liquidity at the end of the day at ECB and obtain the deposit facility rate.

26Values for liquidity are the sum of MRO, marginal refinancing operations (the main instrument of open market operations of the ECB, conducted every week by the ECB with maturity of one week, and LTRO, long term refinancing operations, with one-month, three-months and 3 years maturity. Data on ECB open market operations are available on the ECB website, www.ecb.int.

27There is a heroic assumption here in the matching of data with the model. The model analyzes steady states, while data are daily values. The assumption is thus that one day of transactions is sufficient to reach the steady state of the model. Clearly this is far from realistic, but assuming, for instance, to consider only the daily rate t = 0 days after an open market operation that changes the amount of liquidity provided to the economy does not alter the picture.
different in terms of underlying economic conditions and therefore several factors, rather than just provision of liquidity, could explain the data. To shed more light on the issue we divided the sample in four periods, roughly indicating the pre-financial crisis period (2004-2007), the financial crisis periods (2008-2009), the beginning of the sovereign crisis period in Europe (2010-2011), and finally the period 2012-2013, when the ECB, through its two LTRO operations, flooded the market with liquidity. The first and last periods are those when the rate on the interbank market is, respectively, at its maximum and minimum. During the beginning of the financial crisis (observations in blue in the figure) the ECB started providing more liquidity to the economy, thus pushing down the interest rate on the interbank market. However during the sovereign crisis period (observations in green in the figure) a lower level of liquidity supported roughly the same interest rate on the interbank market. We consider this shift as an increase in volatility in the market, which interpreted through the mechanisms of our model, determined higher haircuts and a lowering of the collateral pool available for the interbank market, thus pushing towards scarce interest bearing asset type of equilibria. One could almost read figure 8 as an empirical counterpart of figure 7.

There are clearly weak connections between the model and the data provided here. For instance, in the model uncertainty comes from the real asset, while in the data increased uncertainty during the sovereign crisis in Europe came from fear of default of

Figure 8: ECB reserves (MRO + LTRO operations) and Eurepo T/N rate - ECB deposit facility rate.
some sovereign bonds. However notice that the Eurepo rate measures transactions on the best collateral, arguably for most (if not all) of the sample, German Bund. Therefore our model economy and data provided could effectively describe a European market in which there is only one (or very few) risk-free bonds and a large class of other risky types of securities used in the interbank market, that in particular during the sovereign crisis periods, included several types of sovereign bonds. As such, the data provided is clearly suggestive evidence of the empirical validity of the model.

5 Conclusion

This paper provided a general equilibrium micro-founded description of the interbank market and the effect of open market operations. We show that, even in a liquidity trap, Central bank can influence quantities exchanged on the interbank market. In this framework we then provided an analysis of the effect of unconventional monetary policies, a change in the composition of the balance sheet and a swelling of the Central bank’s balance sheet. While our analysis has mainly a positive objective, the model here provided can form the basis for normative implications of unconventional monetary policies.

References


Appendix

A Model with only money and bonds

A.1 Formal Derivation of Equilibrium

A.1.1 Equilibrium with $\psi = \frac{\beta}{\gamma}$ and $\gamma > \beta$

In this equilibrium we do not need to impose that $q^{21} = q^{22}$. In fact from (26) we have that $u'(q^{21}) + u'(q^{22}) = 2$ Since $u'(q) \geq 1$ because buyers will never want to consume more than $q^*$, the only way (26) can hold is when $u'(q^{21}) = u'(q^{22}) = 1$. This implies that whenever $\psi = \frac{\beta}{\gamma}$, $q^{21} = q^{22} = q^*$.

Moreover since buyers are already consuming the optimal quantity $u'(q^*) = 1$, from (14) and (21) we derive that $\mu^1 = \mu^2 = \zeta = 0$, that is banks carry excess reserves to the next CM, and that the constraint on the interbank market is slack $Rl < \hat{b}_1$.

From (22), using (20) and $u'(q^{22}) = 1$, we have that $u'(q^{12}) = R + \nu^2$. Instead from (15) we have that $u'(q^{11}) = R - \nu^1$. Substituting for $R$ we have that $u'(q^{12}) - u'(q^{11}) > 0$ and since $u'(\cdot)$ will be at least equal to one, $u'(q^{12}) > 1$ which implies from (20) that $\lambda^2 > 0$. Also from concavity of utility we will have that $q^{12} < q^{11}$ or $\frac{\hat{m}^2}{\rho - \varepsilon} < \frac{\hat{m}^1}{\rho + \varepsilon}$. But then since $\lambda^2 > 0$ and we are assuming that $l = d = 0$ the previous inequality becomes

$$\frac{\hat{m}^2}{\rho - \varepsilon} = \frac{m}{\rho - \varepsilon} < \frac{\hat{m}^1}{\rho + \varepsilon} \leq \frac{m}{\rho + \varepsilon}$$

(where the last inequality comes from $\hat{m}^1 \leq m$) which is a contradiction. Therefore banks must be effectively using the interbank market, $l = d > 0$, and $\nu^1 = \nu^2 = 0$.

Now from (15) and (22) we obtain

$$u'\left(\frac{\hat{m}^1 + l}{\rho + \varepsilon}\right) = R, \quad u'\left(\frac{\hat{m}^2}{\rho - \varepsilon}\right) = R,$$

which substituted in (13) and (20), and using $u'(q^{21}) = u'(q^{22}) = 1$, gives us $R = 1 + \lambda^1$ and $R = 1 + \lambda^2$. Since by proposition 2, $R = 1/\psi = \gamma/\beta > 1$, we have that both $\lambda^1$ and $\lambda^2$ are positive.
Since we are looking for standard contracts where \( q \) that \( \psi < \), we have that \( u'(q^{11}) = u'(q^{12}) \). This implies that we do not need to impose the condition for standard contract \( q^{11} = q^{12} \). Using the definitions of \( q^{11} \) and \( q^{12} \)

\[
q^{11} = \frac{m + l}{\rho + \varepsilon}, \quad q^{12} = \frac{m - l}{\rho - \varepsilon}.
\]

\( l \) must be necessarily \( l = \frac{\varepsilon}{\rho} m \), therefore \( q^{11} = q^{12} = \frac{m}{\rho} \), where from equation (25), \( m \) satisfies

\[
\frac{\gamma}{\beta} = u'\left( \frac{m}{\rho} \right).
\] (41)

For this equilibrium to exist it must therefore be that \( \delta \) is low enough such that in the market there is plentiful of assets. In fact the equilibrium implies that \( Rl < \hat{b}^1 \) and that \( \hat{b}^2 > 0 \) or differently

\[
\frac{b - Rl}{1 - \rho - \varepsilon} > \lambda^* \quad \text{and} \quad \frac{b + Rl}{1 - \rho + \varepsilon} > \lambda^*.
\]

Using the monetary policy rule (4) \( b = \left( \frac{1}{\gamma} - 1 \right) m \) and (41) the previous inequalities imply that:

\[
\delta < \frac{m}{(1 - \rho - \varepsilon)q^* + (1 + \frac{\varepsilon}{\beta}) m} \quad \text{and} \quad \delta < \frac{m}{(1 - \rho + \varepsilon)q^* + (1 - \frac{\varepsilon}{\beta}) m}.
\]

**A.1.2 Equilibrium with \( \frac{\beta}{\gamma} < \psi < 1 \) and \( \gamma > \beta \)**

Consider equation (26): imposing \( q^{21} = q^{22} \) it must be that \( \psi^2 = u'(q^{21}) \), but since \( \frac{\beta}{\gamma} < \psi < 1 \) we have that \( 1 < \psi^2 = u'(q^{21}) \), that is \( q^{21} = q^{22} < q^* \). From (21) we then have \( \mu^2 > 0 \).

Now consider equation (14). Since buyers are consuming less than the optimal quantity, at least one between \( \lambda^1 \) and \( \zeta \) must be greater than zero. Suppose \( \zeta \geq 0 \) (\( Rl \leq \hat{b}^1 \)) and \( \mu^1 > 0 \) (\( \hat{b}^1 = 0 \)).

This implies that \( l = 0 \) and \( \nu^1 > 0 \) and therefore \( q^{11} = \frac{\hat{m}^1}{\rho + \varepsilon} \), \( q^{12} = \frac{\hat{m}^2}{\rho - \varepsilon} \). From (25) and (26), since \( \psi < 1 \) and we have imposed \( q^{21} = q^{22} \), at least one between \( \lambda^1 \) and \( \lambda^2 \) must be positive. Suppose \( \lambda^1 = 0 \) and \( \lambda^2 > 0 \). This implies \( \hat{m}^1 < m \) and, since \( d = l = 0 \), \( \hat{m}^2 = m \). But then, for any \( \varepsilon \), it will never be possible that \( q^{11} = q^{12} \) since \( q^{11} = \frac{\hat{m}^1}{\rho + \varepsilon} < \frac{m}{\rho + \varepsilon} < \frac{\hat{m}^2}{\rho - \varepsilon} = q^{12} \). Suppose instead that \( \lambda^1 > 0 \) and \( \lambda^2 = 0 \). From (13) we have \( u'(d^{11}) = u'(d^{21}) + \lambda^1 \) and from (20) \( u'(d^{12}) = u'(d^{22}) \).

Since we are looking for standard contracts where \( q^{21} = q^{22} \) and \( q^{11} = q^{12} \), \( \lambda^1 > 0 \) and \( \lambda^2 = 0 \) will never imply such contracts. Therefore both \( \lambda^1 > 0 \) and \( \lambda^2 > 0 \).

But then \( \lambda^1 > 0 \) and \( \lambda^2 > 0 \) imply that \( \hat{m}^1 = m \) and \( \hat{m}^2 = m \) (since \( d = 0 \)) and so \( q^{11} = \frac{m}{\rho + \varepsilon} < \frac{m}{\rho - \varepsilon} = q^{12} \), which violates again our standard contract. Therefore it must be the case that \( \zeta > 0 \) (\( Rl = \hat{b}^1 \)) and \( \mu^1 > 0 \) (\( \hat{b}^1 = 0 \)).

The binding constraint for the interbank market implies that \( l > 0 \) (\( \nu^1 = 0 \)) and, since \( l = d \), \( d > 0 \) (\( \nu^2 = 0 \)). From (14) and (15), substituting out for \( \zeta \) we have that,

\[
u'(\frac{\hat{m}^1 + l}{\rho + \varepsilon}) = R\nu'(\frac{b - \hat{b}^1 + m - \hat{m}^1}{1 - \rho - \varepsilon}).
\] (42)
and from (20) and (22) substituting out for $\lambda^2$ obtains

$$u'(\frac{\hat{m}^2}{\rho - \varepsilon}) = Ru'(\frac{b - \hat{b}^2 + m - \hat{m}^2 - d + Rd}{1 - \rho + \varepsilon}).$$

(43)

Since $R > 1$ and $u'(q^{21}) = u'(q^{22}) > 1$, then by the previous equations $u'(q^{11}) > u'(q^{21})$ and $u'(q^{12}) > u'(q^{22})$, which from (13) and (20) imply that both $\lambda^1 > 0$ ($\hat{m}^1 = m$) and $\lambda^2 > 0$ ($\hat{m}^2 + d = m$). Therefore by imposing the equality $q^{11} = q^{12}$ and using $l = d$ we can solve for the optimal amount of money exchanged on the interbank market:

$$\frac{m + l}{\rho + \varepsilon} = \frac{m - l}{\rho - \varepsilon} \implies l = \frac{\varepsilon - m}{\rho}$$

From (25), using (13) and (20), we find the equilibrium amount of money:

$$\frac{\gamma}{\beta} = u'(\frac{m}{\rho})$$

The final object to find is the interest rate on the interbank market $R$. This must be such that $q^{21} = q^{22}$. Given that $RL = \hat{b}^1, \hat{b}^2 = 0$, $\hat{m}^1 = m, \hat{m}^2 = m, \hat{m}^2 + l = m$, imposing the standard contract gives:

$$\frac{b - RL}{1 - \rho - \varepsilon} = \frac{b + RL}{1 - \rho + \varepsilon}$$

Using monetary policy $b = (\frac{1}{\delta} - 1)m$ and $l = \frac{\varepsilon}{\rho}m$ we finally obtain:

$$R = (\frac{1}{\delta} - 1) \frac{\rho}{1 - \rho}$$

(44)

The existence of this equilibrium requires $R > 1$ which from the previous implies $\delta < \rho$. Therefore this equilibrium exists for $\gamma > \beta$ and $\hat{\delta} \leq \delta < \rho$ where lemma 2 in the text indeed shows that $\hat{\delta} < \rho$ for $\gamma > \beta$.

A.1.3 Equilibrium with $\psi = 1$ and $\gamma > \beta$

Since $\psi = 1$ money and government bonds are equivalent. From (25) and (26) this implies that $\lambda_1 = \lambda_2 = 0$. Therefore, from (13) $q^{11} = q^{21}$ and from (20) $q^{12} = q^{22}$. From (21), $q^{22} < q^*$ implies $\mu^2 > 0$ and therefore $\hat{b}^2 = 0$.

We consider first the case in which there is no interbank market. Since $q^{12} = q^{22}$, we have

$$\frac{\hat{m}^2}{\rho - \varepsilon} = \frac{b + m - \hat{m}^2}{1 - \rho + \varepsilon}$$

$$(1 - \rho + \varepsilon)\hat{m}^2 = (\rho - \varepsilon)\left(\frac{1}{\delta} - 1\right)m + m - \hat{m}^2$$

$$\hat{m}^2 = (\rho - \varepsilon)\frac{m}{\delta}$$

that implies $q^{12} = \frac{m_0}{\delta}$. From (14), $q^{21} < q^*$ implies that $\mu^1 > 0$ or $\zeta > 0$ or both. Since there is no
interbank market, $\mu^1 > 0$ and $\hat{b}^1 = 0$. Therefore, using $q^{11} = q^{21}$

$$\frac{\hat{m}^1}{\rho + \varepsilon} = \frac{b + m - \hat{m}^1}{1 - \rho - \varepsilon}$$

$$(1 - \rho - \varepsilon)\hat{m}^1 = (\rho + \varepsilon) \left( \left( \frac{1}{\delta} - 1 \right) m + m - \hat{m}^1 \right)$$

$$\hat{m}^1 = (\rho + \varepsilon) \frac{m}{\delta}$$

that implies $q^{11} = \frac{m}{\delta}$. In this equilibrium $q^{11} = q^{12} = \frac{m}{\delta}$ without imposing any exogenous restriction. Obviously $q^{21} = q^{22}$ and given $\gamma > \beta$, from (26) we have also that $q^{21} = q^{22} < q^*$. The existence of this equilibrium requires $\hat{m}^1 \leq m$ and $\hat{m}^2 \leq m$. Given $\hat{m}^1 > \hat{m}^2$, we need to check that $\hat{m}^1 \leq m$

$$\hat{m}^1 \leq m$$

$$(\rho + \varepsilon) \frac{m}{\delta} \leq m$$

$$\delta \geq \rho + \varepsilon$$

(45)

This equilibrium exists for $\delta \geq \rho + \varepsilon$ and using (25), $m$ is the solution to

$$\frac{\gamma}{\beta} = u' \left( \frac{m}{\delta} \right)$$

Therefore, for $\delta < \rho + \varepsilon$ if $l = d = 0$ we have $q^{11} \neq q^{12}$ and $q^{11} < q^{21}$.

For $\delta < \rho + \varepsilon$ we require $l = d > 0$ and $\mu^1 = \nu^2 = 0$. Therefore, given $q^{21} < q^*$, from (14) $\mu^1 = 0$ and $\zeta > 0$. Substituting (14) in (15), given (13) we have that $R$ must be equal to one. The same result is obtained substituting (20) in (22).

We assume that when a bank is indifferent between reducing his amount of borrowing or keeping excesses reserves she prefers to reduce his borrowing. This allow us, using $q^{11} = q^{21}$, to derive the amount of borrowing $l$

$$\frac{m + l}{\rho + \varepsilon} = \frac{b - l}{1 - \rho - \varepsilon}$$

$$l = (\rho + \varepsilon)b - (1 - \rho - \varepsilon)m$$

$$l = (\rho + \varepsilon) \left( \frac{1}{\delta} - 1 \right) m - (1 - \rho - \varepsilon)m$$

$$l = \frac{\rho + \varepsilon - \delta}{\delta} m < \frac{\varepsilon}{\rho} m$$

and therefore

$$q^{11} = \frac{m + l}{\rho + \varepsilon} = \frac{m + \frac{\varepsilon + \varepsilon - \delta}{\delta} m}{\rho + \varepsilon} = \frac{m}{\delta} = q^{12}$$

In this equilibrium $q^{11} = q^{12} = \frac{m}{\delta}$ without imposing any exogenous restriction. Obviously $q^{21} = q^{22}$ and given $\gamma > \beta$, from (26) we have also that $q^{21} = q^{22} < q^*$. Using (25), $m$ is the solution to

$$\frac{\gamma}{\beta} = u' \left( \frac{m}{\delta} \right)$$

40
This equilibrium requires $\hat{m}^2 + l \leq m$. Since $q^{12} = q^{22}$, we derived above that $\hat{m}^2 = (\rho - \varepsilon) \frac{m}{\delta}$ and we have\(^{28}\)

$$\begin{align*}
\hat{m}^2 + l &\leq m \\
(\rho - \varepsilon) \frac{m}{\delta} + \frac{\rho + \varepsilon - \delta}{\delta} m &\leq m \\
\frac{(\rho - \varepsilon)}{\delta} + \frac{(\rho + \varepsilon)}{\delta} &\leq 2 \\
\delta &\geq \rho
\end{align*}$$

Therefore the existence of an equilibrium with $\psi = 1$ requires $\delta \geq \rho$. For $\delta < \rho$ the interbank market exists and the amount of borrowing is decreasing in $\delta$. For $\delta \geq \rho$ also banks of type 1 start keeping excess reserves and therefore the interbank market ceases to exist.

### A.2 Proofs

**Proof of proposition 2 (Equilibrium prices).** The first part of the proposition comes from the inspection of (25) and (26). The optimum consumption for a buyer is $q^*$, such that $u'(q^*) = 1$. From (26), if $\psi < \frac{\beta}{\gamma}$ the LHS will be less than one, while the RHS can never be less than one since buyers will never consume more than the optimal quantity $q^*$. This would imply an infinite demand for the government bond, but this cannot be an equilibrium.

Now substitute for $\frac{\gamma}{\beta}$ in (25) using (26). Rearranging we obtain:

$$
\left( \frac{1}{\psi} - 1 \right) \left[ u' \left( \frac{b - \hat{b}_1 + m - \hat{m}_1}{1 - \rho - \varepsilon} \right) + u' \left( \frac{b - \hat{b}_2 + m - \hat{m}_2 - d + Rd}{1 - \rho + \varepsilon} \right) \right] = \lambda_1 + \lambda_2
$$

If $\psi > 1$ the LHS is negative, but this implies that at least one between the Lagrange multipliers $\lambda_1$ and $\lambda_2$ must be negative, which is impossible. Therefore $\psi \leq 1$.

The second part of the proposition comes from the equilibrium on the asset market. The existence of the interbank market implies that $l = d > 0$, therefore both $\nu^1 = \nu^2 = 0$, and that $\hat{b}^1 > 0$, implying $\mu^1 = 0$. From (15) using (14) to substitute for $\zeta$, and from (22) using (20) to substitute for $\lambda^2$, we get

$$
\begin{align*}
&u' \left( \frac{\hat{m}_1 + l}{\rho + \varepsilon} \right) = Ru' \left( \frac{b - \hat{b}_1 + m - \hat{m}_1}{1 - \rho - \varepsilon} \right) \\
&u' \left( \frac{\hat{m}_2}{\rho - \varepsilon} \right) = Ru' \left( \frac{b - \hat{b}_2 + m - \hat{m}_2 - d + Rd}{1 - \rho + \varepsilon} \right)
\end{align*}
$$

Substituting the previous expressions in (26) we find

$$
Ru' \frac{\gamma}{\beta} = \left\{ \frac{1}{2} u' \left( \frac{\hat{m}_1 + l}{\rho + \varepsilon} \right) + \frac{1}{2} u' \left( \frac{\hat{m}_2}{\rho - \varepsilon} \right) \right\}
$$

\(^{28}\)When $R = 1$ the determination of $\hat{m}_2$ is not affected by the amount of lending in the interbank market.
From (25), using (13) and (20) we obtain

$$\frac{\gamma}{\beta} = \left\{ \frac{1}{2} u' \left( \frac{\tilde{m}^1 + l}{\rho + \varepsilon} \right) + \frac{1}{2} u' \left( \frac{\tilde{m}^2}{\rho - \varepsilon} \right) \right\}$$ (49)

Therefore $R = \frac{1}{\psi}$. □

**Proof of lemma 1 (Thresholds ordering).** Comparing the two disequalities, we derive that their ordering depends on:

$$\frac{\gamma m}{\beta \rho} \gg q^*$$

and for $\gamma = \beta$ this condition is satisfied with equality since from (41): $u' \left( \frac{m}{\rho} \right) = \frac{\gamma}{\beta} = 1 = u'(q^*)$.

We want to determine what happens to the LHS when $\gamma$ increases. Taking the derivative with respect to $\gamma$ we have

$$m(\gamma, \beta, \rho) + \frac{\gamma}{\beta} \cdot \frac{\partial m(\gamma, \beta, \rho)}{\partial \gamma} \gg 0$$

From (41) using the implicit function theorem we have

$$\frac{\partial m(\gamma, \beta, \rho)}{\partial \gamma} = -\frac{1}{u'' \left( \frac{m}{\rho} \right)} \cdot \frac{\rho}{\beta} = \frac{\rho}{\beta} \cdot \frac{m}{\rho} < 0$$

Using this result:

$$\frac{m(\gamma, \beta, \rho)}{\beta} + \frac{\gamma}{\beta} \cdot \frac{\partial m(\gamma, \beta, \rho)}{\partial \gamma} \gg 0 \Rightarrow m(\gamma, \beta, \rho) + \frac{\gamma}{\beta} \cdot \frac{\partial m(\gamma, \beta, \rho)}{\partial \gamma} \gg 0 \Rightarrow u'' \left( \frac{m}{\rho} \right) m(\gamma, \beta, \rho) + \frac{\gamma}{\beta} \rho \gg 0$$

$$\Leftrightarrow u'' \left( \frac{m}{\rho} \right) \frac{m}{\rho} \gg -1 \Leftrightarrow \frac{u'' \left( \frac{m}{\rho} \right) m}{u'' \left( \frac{m}{\rho} \right) \rho} \gg 1$$

where in the fourth passage we used again (41). Therefore, a condition on the Arrow-Pratt coefficient determines which of the two constraint is binding. □

**Proof of lemma 2 (Threshold less than $\rho$).** We will only show the case when $-\frac{u''(x)x}{u(x)} > 1$, so that $\delta = \frac{m}{(1-\rho-\varepsilon)q^* + (1 + \frac{\gamma \varepsilon}{\beta \rho})m}$. The proof when $-\frac{u''(x)x}{u(x)} < 1$ is similar. We want to show that:

$$\frac{m}{(1-\rho-\varepsilon)q^* + (1 + \frac{\gamma \varepsilon}{\beta \rho})m} \leq \rho$$

with equality when $\gamma = \beta$. Rearranging the equation we have:

$$\frac{m}{\rho} \left( 1 - \rho - \frac{\gamma}{\beta} \varepsilon \right) \leq (1-\rho-\varepsilon)q^*$$

clearly if $\gamma = \beta$ then $\frac{m}{\rho} \leq q^*$. But since $u' \left( \frac{m}{\rho} \right) = \frac{\gamma}{\beta} = 1 = u'(q^*)$ then $\frac{m}{\rho} = q^*$ and the condition holds with equality.

Suppose now we increase $\gamma$. By taking the derivative with respect to $\gamma$ of the condition we
have that:

$$\frac{\partial m}{\partial \gamma} \left(1 - \rho - \frac{\gamma}{\beta} \right) - \frac{m \varepsilon}{\rho \beta} \leq 0$$

and by using the implicit function theorem on (41) to substitute for $\frac{\partial m}{\partial \gamma}$ and rearranging we obtain

$$-\frac{u''(m/\rho)}{u'(m/\rho)} \frac{m}{\rho} \geq 1 - \frac{1 - \rho}{u'(m/\rho)}$$

Since $\frac{1 - \rho}{u'(m/\rho)} > 0$ and we assumed that $-\frac{u''(x)x}{u'(x)} > 1$, the condition always holds strictly.

Proof of lemma 3 (Discontinuity at $R(\delta)$). We know that for $\delta \geq \hat{\delta}$ we have that $R(\delta) = (\frac{1}{\beta} - 1) \frac{\rho}{1 - \rho}$, while for $\delta < \hat{\delta}$ $R = \frac{\gamma}{\beta}$. We focus on the difference $R(\delta) - \frac{\gamma}{\beta}$. Assume that $-\frac{u''(x)x}{u'(x)} > 1$, then from lemma 1 we know that $\delta = \frac{m}{(1 - \rho - \varepsilon)q^* + (1 + \frac{\gamma}{\beta} m)}$, therefore:

$$R(\delta) - \frac{\gamma}{\beta} = \frac{1 - \rho - \varepsilon}{1 - \rho} \left[ q^* \frac{\rho}{m} - \frac{\gamma}{\beta} \right]$$

and in particular $\text{sign} \left( R(\delta) - \frac{\gamma}{\beta} \right) = \text{sign} \left( q^* - \frac{\gamma}{\beta} \frac{m}{\rho} \right)$. From lemma 1 we know that $-\frac{u''(x)x}{u'(x)} > 1$ implies $q^* - \frac{\gamma}{\beta} \frac{m}{\rho} < 0$, giving the statement.

B Model with short lived asset

B.1 Type 1 Bank: Problem and FOC

Bank 1 solves the following maximization problem:

$$\max_{\hat{m}, \hat{b}, \hat{a}, \lambda, l} \left( \rho + \varepsilon \right) u \left( \frac{\hat{m} + l}{\rho + \varepsilon} \right) + \left( 1 - \rho - \varepsilon \right) u \left( \frac{b - \hat{b} + k_l (a - \hat{a}) + m - \hat{m} + \bar{k} \hat{a} - R l}{1 - \rho - \varepsilon} \right) + \bar{k} (a - \hat{a}) - k_l (a - \hat{a}) + (\hat{b} + \bar{k} \hat{a} - R l)$$

s.t. $\hat{a} \geq 0 (\xi)$, $\hat{b} \geq 0 (\mu)$, $l \geq 0 (\nu)$

$$a \geq \hat{a} (\theta)$, $b \geq \hat{b} (\omega)$, $m \geq \hat{m} (\lambda)$

$$R l \leq \hat{b} + k_l \hat{a} (\zeta)$$

43
where in parenthesis we put the associated Lagrange multiplier. The first order conditions

\[
    u'(\frac{\hat{m}^1 + l}{\rho + \varepsilon}) = u'\left(\frac{b - \hat{b}^1 + k_1(a - \hat{a}^1) + m - \hat{m}^1}{1 - \rho - \varepsilon}\right) + \lambda^1
\]

\[
    u'(\frac{b - \hat{b}^1 + k_1(a - \hat{a}^1) + m - \hat{m}^1}{1 - \rho - \varepsilon}) = 1 + \mu^1 - \omega^1 + \zeta^1
\]

\[
    u'(\frac{b - \hat{b}^1 + k_1(a - \hat{a}^1) + m - \hat{m}^1}{1 - \rho - \varepsilon}) k_l = k_l(1 + \zeta^1) + \zeta^1 - \theta^1
\]

\[
    u'(\frac{\hat{m}^1 + l}{\rho + \varepsilon}) + \nu^1 = R(1 + \zeta^1)
\]

B.2 Type 2 Bank: Problem and FOC

Bank 2 solves the following maximization problem:

\[
    \max_{\hat{m}^2, \hat{b}^2, \hat{a}^2, d} (\rho - \varepsilon)u\left(\frac{\hat{m}^2}{\rho - \varepsilon}\right) + (1 - \rho + \varepsilon)u\left(\frac{b - \hat{b}^2 + k_1(a - \hat{a}^2) + m - \hat{m}^2 - d + Rd}{1 - \rho + \varepsilon}\right) + \frac{\hat{k}(a - \hat{a}^2) - k_l(a - \hat{a}^2) + (\hat{b}^2 + \hat{k}\hat{a}^2)}{1 - \rho - \varepsilon}
\]

s.t. \(\hat{a}^2 \geq 0 (\zeta^2), \hat{b}^2 \geq 0 (\mu^2), d \geq 0 (\nu^2)\)

\[a \geq \hat{a}^2 (\theta^2), b \geq \hat{b}^2 (\omega^2), m \geq \hat{m}^2 + d (\lambda^2)\]

where in parenthesis we put the associated Lagrange multiplier. The first order conditions

\[
    u'(\frac{\hat{m}^2}{\rho - \varepsilon}) = u'\left(\frac{b - \hat{b}^2 + k_1(a - \hat{a}^2) + m - \hat{m}^2 - d + Rd}{1 - \rho + \varepsilon}\right) + \lambda^2
\]

\[
    u'(\frac{b - \hat{b}^2 + k_1(a - \hat{a}^2) + m - \hat{m}^2 - d + Rd}{1 - \rho + \varepsilon}) = 1 + \mu^2 - \omega^2
\]

\[
    u'(\frac{b - \hat{b}^2 + k_1(a - \hat{a}^2) + m - \hat{m}^2 - d + Rd}{1 - \rho + \varepsilon}) k_l = k_l(1 + \zeta^2 - \theta^2)
\]

\[
    (R - 1)u'(\frac{b - \hat{b}^2 + k_1(a - \hat{a}^2) + m - \hat{m}^2 - d + Rd}{1 - \rho + \varepsilon}) + \nu^2 = \lambda^2
\]

B.2 Type 2 Bank: Problem and FOC
B.3 FOC of Centralized Market

We can then define the value function of the decentralized market as:

\[
\mathbb{E}V(m, b, a) = \frac{1}{2} \left[ (\rho + \varepsilon)u(q^{11}) + (1 - \rho - \varepsilon)u(q^{21}) + (\bar{k} - k_1)(a - \bar{a}^1) + (k\bar{a}^1 + \bar{b}^1 - Rl) \right] +
\]
\[
+ \frac{1}{2} \left[ (\rho - \varepsilon)u(q^{12}) + (1 - \rho + \varepsilon)u(q^{22}) + (\bar{k} - k_1)(a - \bar{a}^2) + (k\bar{a}^2 + \bar{b}^2) \right]
\]

Taking first order conditions with respect to \(m\), \(b\) and \(a\), and considering the constraints that the bank will face in the DM we have:

\[
\frac{\gamma}{\beta} = \frac{1}{2} \left[ u' \left( \frac{b - \bar{b}^1 + k_1(a - \bar{a}^1) + m - \bar{m}^1}{1 - \rho - \varepsilon} \right) + \lambda^1 \right] +
\]
\[
+ \frac{1}{2} \left[ u' \left( \frac{b - \bar{b}^1 + k_1(a - \bar{a}^1) + m - \bar{m}^1}{1 - \rho - \varepsilon} \right) + \lambda^2 \right] \tag{58}
\]

\[
\frac{\psi \gamma}{\beta} = \frac{1}{2} \left[ u' \left( \frac{b - \bar{b}^1 + k_1(a - \bar{a}^1) + m - \bar{m}^1}{1 - \rho - \varepsilon} \right) + \omega^1 \right] +
\]
\[
+ \frac{1}{2} \left[ u' \left( \frac{b - \bar{b}^1 + k_1(a - \bar{a}^1) + m - \bar{m}^1}{1 - \rho - \varepsilon} \right) + \omega^2 \right] \tag{59}
\]

\[
\frac{p}{\beta} = \frac{1}{2} \left[ u' \left( \frac{b - \bar{b}^1 + k_1(a - \bar{a}^1) + m - \bar{m}^1}{1 - \rho - \varepsilon} \right) k_1 + \bar{k} - k_1 + \theta^1 \right] +
\]
\[
+ \frac{1}{2} \left[ u' \left( \frac{b - \bar{b}^1 + k_1(a - \bar{a}^1) + m - \bar{m}^1}{1 - \rho - \varepsilon} \right) k_1 + \bar{k} - k_1 + \theta^1 \right] \tag{60}
\]

B.4 Formal derivation of equilibrium

B.4.1 Equilibrium with \(\psi = \frac{\beta}{\gamma}\) and \(p = \beta\bar{k}\)

When \(\psi = \frac{\beta}{\gamma}\), from no arbitrage \(p = \beta\bar{k}\). In this equilibrium \(\omega^1 = \omega^2 = \theta^1 = \theta^2 = 0\) and from the euler equations (59) and (60) we have \(u' (q^{21}) = u' (q^{22}) = 1\), that implies \(q^{21} = q^{22} = q^\ast\). From (51), (55), (52) and (56), this also implies \(\mu^1 = \mu^2 = z^1 = z^2 = \zeta = 0\), or \(\bar{b} \succ 0, \bar{b}^2 \succ 0, \bar{a}^1 \succ 0, \bar{a}^2 \succ 0\) and \(Rl < \bar{b}^1 + k_1\bar{a}^1\).

Using (54) and (57), we find \(u' (q^{12}) = R + \nu^2\), while substituting (51) and (53) we get \(u' (q^{11}) = R - \nu^1\). Imposing \(q^{11} = q^{12}\), necessarily \(\nu^2 = \nu^1\) and this can be satisfied only if \(\nu^1 = \nu^2 = 0\). Therefore \(l = d > 0\). The first order conditions (53) and (57) become

\[
u' \left( \frac{\bar{m}^1 + l}{\rho + \varepsilon} \right) = R, \quad \nu' \left( \frac{\bar{m}^2}{\rho - \varepsilon} \right) = R,
\]

which substituted in (50) and (54), using \(u' (q^{21}) = u' (q^{22}) = 1\), give us \(R = 1 + \lambda_1\) and \(R = 1 + \lambda_2\). Since from Lemma 2 \(R = \frac{1}{\psi} = \frac{\beta}{\gamma} > 1\), we have \(\lambda_1 = \lambda_2 > 0\), that implies \(\bar{m}^1 = m\) and
\[ \dot{m}^2 = m - d = m - l. \]  
Therefore, we have

\[ q^{11} = \frac{m + l}{\rho + \varepsilon}, \quad q^{12} = \frac{m - l}{\rho + \varepsilon}, \]

and given \( q^{11} = q^{22}, \) \( l \) must be necessarily equal to \( \frac{\varepsilon}{\rho} m. \) Therefore, \( q^{11} = q^{12} = \frac{m}{\rho} \) and from (58) the equilibrium value of \( m \) satisfies

\[ u'(\frac{m}{\rho}) = \frac{\gamma}{\beta} \]

(61)

The existence of this equilibrium requires that there are enough interest bearing assets such that \( q^{21} = q^{22} = \hat{q}^* \) and \( l \) is such that \( q^{11} = q^{12}. \) The equilibrium implies \( Rl < \hat{\pi}^1 \) and \( \hat{\pi}^2, \) or

\[ \frac{\pi - Rl}{1 - \rho - \varepsilon} > \hat{q}^* \quad \text{and} \quad \frac{\pi + Rl}{1 - \rho + \varepsilon} > \hat{q}^*. \]

Using the monetary policy rule \( b = m \left( \frac{1}{2} - 1 \right), \) the market clearing condition for the Lucas tree and the equilibrium values of \( R \) and \( l \) the previous inequalities can be rewritten as

\[
\frac{k_1 A + \left( \frac{1}{2} - 1 \right) m - \frac{\varepsilon}{\beta} \hat{q}^* m}{1 - \rho - \varepsilon} > \hat{q}^* \quad \text{and} \quad \frac{k_1 A + \left( \frac{1}{2} - 1 \right) m + \frac{\varepsilon}{\beta} \hat{q}^* m}{1 - \rho + \varepsilon} > \hat{q}^*.
\]

Here and in the rest of the paper we concentrate on equilibria in which the government is a net debtor of the private sector, \( B_t \geq 0 \ \forall t. \) This implies \( \delta \in (0, 1). \)

When \( A \geq \bar{A}, \) where \( A \) is defined in Proposition 4, central bank intervention throught \( \delta \) is irrelevant. Only \( \gamma \) affects the equilibrium and depending on the coefficient of relative risk aversion also \( \bar{A}. \) However, for \( A < \bar{A} \) the existence of this equilibrium requires \( \delta < 1. \) Differently from the case in which there are only money and government bonds, adding a real asset and considering only equilibria with \( \delta \in (0, 1), \) it is possible the absence of a liquidity trap equilibrium. Williamson shows that a liquidity trap can still exists if we let \( \delta \in (-\infty, \infty). \)

**Lemma 5** Suppose \( A < \bar{A}. \) An equilibrium with \( \psi = \frac{\bar{p}}{\bar{\gamma}} \) and \( p = \beta \bar{k} \) exists if \( \delta \leq \hat{\delta}_1, \) where \( \hat{\delta}_1 \) is defined as

\[
\hat{\delta}_1 \equiv \min \left\{ \frac{m}{(1 - \rho - \varepsilon)q^* + \left( \frac{\varepsilon}{\beta} \right) m - k_1 A}, \frac{m}{(1 - \rho + \varepsilon)q^* + \left( 1 - \frac{\varepsilon}{\beta} \right) m - k_1 A} \right\}
\]

(62)

Given that the amount of \( A \) is relevant to determine the existence of the equilibria found in the case without real asset, we now look at the liquidity trap equilibrium.

**B.4.2 Equilibrium with \( \psi = 1 \) and \( p = \beta \bar{k} + k_1 (\gamma - \beta) \)**

Since \( \psi = 1, \) money and government bonds are equivalent. From (55) we know that for \( \gamma > \beta, \) since \( q^{22} < q^* \) necessarily \( \omega^2 \) is equal to zero (and also \( \theta^2 = 0). \) We can have at most \( \omega^1 > 0, \) but this implies from (59) and (58) that at least \( \lambda_1 > 0; \) therefore \( q^{21} = 0, \) but given the Inada conditions this impossible and consequently \( \omega^1 = 0 \) (and also \( \theta^1 = 0) \) and \( \lambda_1 = \lambda_2 = 0. \) Therefore, from (50) we have \( q^{11} = q^{21} \) and from (54) we have \( q^{12} = q^{22}. \) From (55) and (56), \( q^{22} < q^* \) implies \( \mu^2 \) and \( z^2 \) both greater than zero and therefore \( b^2 = \bar{a}^2 = 0. \)
We consider first the case in which there is no interbank market. Since \( q_{12} = q_{22} \), we have

\[
\hat{m}^2 \rho - \varepsilon = \pi + m - \hat{m}^2 (\rho - \varepsilon) (\pi + m) = \pi + (\rho - \varepsilon) \left( \frac{m}{\delta} + k_tA \right)
\]

that implies \( q_{12} = \frac{m}{\delta} + k_tA \). From (51), \( q_{21} < q^* \) implies \( \mu^1 > 0 \) or \( \zeta > 0 \) or both. Since there is no interbank market, \( \mu^1 > 0 \) and \( \hat{b}^1 = 0 \). From (52), we have \( z^1 > 0 \) and \( \hat{a}^1 = 0 \). Therefore, using \( q_{11} = q_{21} \) we have

\[
\hat{m}^1 \rho + \varepsilon = \pi + m - \hat{m}^1 (\rho + \varepsilon) = \pi + (\rho + \varepsilon) \left( \frac{m}{\delta} + k_tA \right)
\]

that implies \( q_{11} = \frac{m}{\delta} + k_tA \). In this equilibrium, \( q_{11} = q_{12} = q_{22} < q^* \). From (51), \( q_{21} < q^* \) implies \( \mu^1 > 0 \) or \( \zeta > 0 \) or both. Since there is no interbank market, \( \mu^1 > 0 \) and \( \hat{b}^1 = 0 \). Using the same rationale for (52), we have \( z^1 > 0 \) and \( \hat{a}^1 = 0 \). Therefore, using \( q_{11} = q_{21} \) we have

\[
\hat{m}^1 \rho + \varepsilon = \pi + m - \hat{m}^1 (\rho + \varepsilon) = \pi + (\rho + \varepsilon) \left( \frac{m}{\delta} + k_tA \right)
\]

that implies \( q_{11} = \frac{m}{\delta} + k_tA \). In this equilibrium, \( q_{11} = q_{12} = q_{22} < q^* \).

The existence of this equilibrium requires \( \hat{m}^1 \leq m, \hat{m}^2 \leq m \) and \( m > 0 \). From (58) we can write

\[
\frac{\gamma}{\beta} = u' \left( \frac{m}{\delta} + k_tA \right)
\]

and because \( \delta \in (0, 1] \) for \( A \) excessively high, we can have no positive solution for \( m \). Define at this moment the solution for \( m \) as \( m(\delta, \gamma, A) \) and assume it is positive. Given \( \hat{m}^1 > \hat{m}^2 \), we need to check that \( \hat{m}^1 \leq m \):

\[
\hat{m}^1 \leq m
\]

\[
(\rho + \varepsilon) \left( \frac{m}{\delta} + k_tA \right) \leq m
\]

\[
\frac{1}{\delta} \leq \frac{1}{\rho + \varepsilon} = \frac{k_tA}{m(\delta, \gamma, A)}
\]

(65)

When \( A = 0 \) we retrieve the previous condition, \( \delta \geq \rho + \varepsilon \). If the amount of real asset is positive, the existence of this equilibrium for \( \delta \in (0, 1] \) requires \( A \) to be lower than \( A_0 \), as defined in Proposition 5. Obviously, we have \( \bar{A} > A_0 \). In fact, from (59) when \( A = \bar{A} \) for \( \delta = 1 \) we have \( \psi = \frac{\delta}{\beta} \). However, in a liquidity trap \( \psi = 1 \), therefore \( \bar{A} > \delta_1 \) if \( A > \bar{A} \) (59) is not satisfied because the LHS is greater than the RHS and this is true for all \( \delta \in (0, 1] \).

We can now move to the case in which an interbank market exists. From the case analyzed before, if \( l = d = 0 \) and \( \delta < \delta^2 \), we would have \( q_{11} = q_{12} = \frac{m}{1 - \rho + \varepsilon} \). Therefore, we require \( l = d > 0 \) and \( \nu^1 = \nu^2 = 0 \). Given \( q_{21} < q^* \), from (51) we need \( \mu^1 = 0 \) (\( z^1 = 0 \)) and \( \zeta > 0 \). Substituting (51) or (52) in (53), given (50) we have that \( R \) must be equal to one. The same result is obtained substituting (54) in (57). In order to avoid equilibrium indeterminacy, we assume that when a bank is indifferent between reducing his amount of borrowing or keeping excesses reserves she prefers to reduce her borrowing. This allow us, using \( q_{11} = q_{21} \), to derive the amount
of borrowing $l$

$$\frac{m + l}{\rho + \varepsilon} = \frac{\pi - l}{1 - \rho - \varepsilon}$$

$$l = (\rho + \varepsilon)\pi - (1 - \rho - \varepsilon)m = (\rho + \varepsilon)k_1A + \left(\frac{\rho + \varepsilon - \delta}{\delta}\right)m$$

and therefore

$$q^{11} = \frac{m + l}{\rho + \varepsilon} = \frac{m + (\rho + \varepsilon)\pi - (1 - \rho - \varepsilon)m}{\rho + \varepsilon} = m + k_1A = q^{12}$$

(66)

In this equilibrium $q^{11} = q^{12} = \frac{m}{\pi} + k_1A$ without imposing any exogenous restriction. Obviously $q^{21} = q^{22}$ and given $\gamma > \beta$, from (26) we have also that $q^{21} = q^{22} < q^*$. This equilibrium requires $\hat{m}^2 + l \leq m$. Since $q^{12} = q^{22}$ we derived above that $\hat{m}^2 = (\rho - \varepsilon)(\pi + m)$ and we have

$$\hat{m}^2 + l \leq m$$

$$(\rho - \varepsilon)(\pi + m) + (\rho + \varepsilon)\pi - (1 - \rho - \varepsilon)m \leq m$$

$$2\rho\pi - (1 - \rho + \varepsilon)m \leq (1 - \rho - \varepsilon)m$$

$$2\rho\pi \leq 2(1 - \rho)m$$

$$\rho k_1A \leq (1 - \rho)m - \rho \left(\frac{1}{\delta} - 1\right)m$$

$$\rho k_1A \leq \frac{\delta - \rho}{\delta} m$$

$$\rho k_1A \leq \frac{\delta - \rho}{\delta} m$$

When $A = 0$ we retrieve the same condition of the model without real asset. Assuming $A < A$, we know that exists a $\delta^2$ in the interval $(\rho + \varepsilon, 1)$ such that

$$\frac{1}{\delta^2} = \frac{1}{\rho + \varepsilon} - \frac{k_1A}{m(\delta^2, \gamma, A)}$$

and therefore, keeping $A$ and $\gamma$ constant,

$$\frac{1}{\delta^2} < \frac{1}{\rho} - \frac{k_1A}{m(\delta^2, \gamma, A)}$$

This implies that exists a $\delta^1 < \delta^2$ such that

$$\frac{1}{\delta^1} = \frac{1}{\rho} - \frac{k_1A}{m(\delta^1, \gamma, A)}$$

where $m(\delta^1, \gamma, A)$ is the solution to

$$\frac{\gamma}{\beta} = u'(\frac{m}{\delta^1} + k_1A)$$

and it is lower than $m(\delta^2, \gamma, A)$. 

48
For $\delta > \delta^1$ the LHS decreases and $m(\delta, \gamma, A)$ increases, therefore the condition is satisfied with a strict inequality. Therefore, an equilibrium with a liquidity trap and an interbank market exists for $\delta \geq \delta^1$. In order to verify that the upper thresholds for the region of this equilibrium is $\delta^2$, we take the expression for $l$ and we take the limit for $l$ that goes to zero:

$$l = (\rho + \varepsilon)k_1A + \left(\frac{\rho + \varepsilon - \delta}{\delta}\right)m$$

$$0 = (\rho + \varepsilon)k_1A + \left(\frac{\rho + \varepsilon - \delta}{\delta}\right)m$$

$$\frac{\rho + \varepsilon}{\delta} - 1 = -(\rho + \varepsilon)\frac{k_1A}{m}$$

$$\frac{1}{\delta} = \frac{-k_1A}{\rho + \varepsilon}$$

(67)

$\delta > \delta^1$ that is exactly the condition we found for $\delta^2$.

It is also interesting to note that the condition for $\delta^1$ can be retrieved putting $l = \frac{\varepsilon}{\rho}m$:

$$\frac{\varepsilon}{\rho}m = (\rho + \varepsilon)k_1A + \left(\frac{\rho + \varepsilon - \delta}{\delta}\right)m$$

$$\frac{\varepsilon}{\rho}m + m - (\rho + \varepsilon)k_1A = \left(\frac{\rho + \varepsilon}{\delta}\right)m$$

$$\frac{\rho + \varepsilon}{\rho}m - (\rho + \varepsilon)\frac{k_1A}{m} = \frac{\rho + \varepsilon}{\delta}$$

$$\frac{1}{\delta} = \frac{-k_1A}{\rho + \varepsilon}$$

and since for a given couple $(\gamma, A)$ $\frac{\varepsilon}{\rho}$ is constant, we can easily see that $l$ is decreasing in $\delta$. Before to conclude, it is important to remark that $A < A$ is a sufficient condition for the existence of this equilibrium, but not necessary. In fact, this equilibrium can exist also for an $A > A$ but sufficiently low such that $\delta^1 < 1$.

### B.4.3 Equilibrium with $\frac{\beta}{\gamma} < \psi < 1$ and $\beta k < p < \beta k + k_1(\gamma - \beta)$

Consider (59). Imposing $q^{21} = q^{22}$ it must be that $\psi_2 = u'(q^{21})$ and $p = \beta k + k_1u'(q^{21}) - 1$. As before, from (55) and (56) we have $\omega^2 = \delta^2 = 0$. Moreover, since $\psi < 1$, from (58) and (59) at least $\lambda_1 > 0$, but therefore if $\omega_1 > 0$ we have $q^{21} = 0$, but from the Inada conditions this is impossible. Therefore $\omega^1 = \theta^1 = 0$. From (59), since $\frac{\beta}{\gamma} < \psi < 1$ necessarily $q^{22} = q^2 < q^1$. From (55) and (56) we then have $\mu^2 > 0$ and $z^2 > 0$.

Now consider (51). Since buyers are consuming less than the optimal quantity, at least one between $\mu^1$ and $\zeta$ must be greater than zero. Suppose $\zeta > 0$ and $\mu^1 > 0$, or $b^1 < 0$. We know from Lemma 1 that in this case also $z^1 > 0$ and $d^1 = 0$. This implies that $l = 0$ and $\nu^1 > 0$ and therefore $q^{11} = \frac{m^1}{\rho + \varepsilon}$ and $q^{12} = \frac{m^2}{\rho + \varepsilon}$. From (58) and (59), since $\psi < 1$ at least one between $\lambda_1$ and $\lambda_2$ must be greater than zero. Suppose $\lambda_1 = 0$ and $\lambda_2 > 0$. This implies $\dot{m}^1 < m$ and, since $d = l = 0$, $\dot{m}^2 = m$. But then, for any $\varepsilon > 0$, it will never be possible that $q^{11} = q^{12}$ since $q^{11} = \frac{m^1}{\rho + \varepsilon} < \frac{m}{\rho + \varepsilon} = q^{12}$. Suppose instead that $\lambda^1 > 0$ and $\lambda^2 = 0$. From (50) we have $u'(q^{11}) = u'(q^{11}) + \lambda^1$ and from (54) $u'(q^{12}) = u'(q^{22})$. Since we are looking for standard contracts where $q^{21} = q^{22}$ and $q^{11} = q^{12}$, $\lambda^1 > 0$ and $\lambda^2 = 0$ will never imply such contracts. Therefore both
\( \lambda_1 > 0 \) and \( \lambda_2 > 0 \).

But then \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \) imply that \( \hat{m}_1 = m \) and, since \( d = 0 \), \( \hat{m}_2 = m \). Therefore \( q^{11} = \frac{m}{\rho + \varepsilon} < \frac{m}{\rho - \varepsilon} = q^{22} \), which violates again the standard contract. Therefore, it must be the case that \( \zeta > 0 \) and \( \mu^1 = \theta^1 = 0 \), then \( \hat{b}_1 > 0 \) and \( \hat{m}_2 > 0 \). The binding constraint for the interbank market implies that \( l > 0 \) (\( \nu^1 = 0 \)) and since \( l = d > 0 \) (\( \nu^2 \)). From (51) and (53) substituting out for \( \zeta \) we have that

\[
\dot{m}_1 \rho + \varepsilon = \frac{\pi + \hat{m}_1 + m - \hat{m}_1}{1 - \rho - \varepsilon}
\]

and from (54) and (57) substituting out of \( \lambda_2 \) we have

\[
\dot{m}_2 \rho - \varepsilon = \frac{\pi + \hat{m}_2 + m - \hat{m}_2 - d + Rd}{1 - \rho + \varepsilon}
\]

Since \( R > 1 \) and \( u'(q^{21}) = u'(q^{22}) > 1 \), then by the previous equations \( u'(q^{11}) > u'(q^{21}) \) and \( u'(q^{12}) > u'(q^{22}) \), that from (50) and (54) implies \( \lambda_1 > 0 \) (\( \hat{m}_1 = m \)) and \( \lambda_2 > 0 \) (\( \hat{m}_2 + d = m \)). Therefore, imposing the equality \( q^{11} = q^{12} \) and using \( l = d \) we can solve for the optimal amount of money exchanged in the interbank market

\[
\frac{m + l}{\rho + \varepsilon} = \frac{m - l}{\rho - \varepsilon} \Rightarrow l = \frac{\varepsilon m}{\rho}
\]

From (58), using (50) and (54) the optimal amount of money is the solution to

\[
\gamma = u'\left(\frac{m}{\rho}\right)
\]

The final object to find is the interbank interest rate \( R \). This must be such that \( q^{21} = q^{22} \). Given that \( Rl = \hat{\pi}^1, \hat{\pi}^2 = 0, \hat{m}_1 = m \) and \( \hat{m}_2 + l = m \), imposing the standard contract gives

\[
\frac{\pi - Rl}{1 - \rho - \varepsilon} = \frac{\pi + Rl}{1 - \rho + \varepsilon}
\]

Using monetary policy \( b = m \left(\frac{\delta}{\beta} - 1\right) \) and \( l = \frac{\varepsilon}{\rho} m \) we finally obtain

\[
R = \frac{\rho}{1 - \rho} \left[\frac{\left(\frac{1}{\delta} - 1\right)}{1 - \rho} + \frac{k_1 A}{m}\right]
\]

The existence of this equilibrium requires \( R > 1 \), which from the previous expression implies

\[
\frac{1}{\delta} > \frac{1}{\rho} - \frac{k_1 A}{m}
\]

This condition is equivalent to the one we found for the equilibrium with \( \psi = 1 \). We need then to show that the threshold for \( \delta \) is the previously found \( \delta^1 \). From the above condition and given \( \frac{\gamma}{\beta} = u'\left(\frac{m}{\rho}\right) \), we can write

\[
\frac{1}{\delta} = \frac{1}{\rho} - \frac{k_1 A}{m} = \frac{1}{\rho} - \frac{k_1 A}{\rho \Omega}
\]

50
where $\Gamma \equiv (u')^{-1}\left(\frac{z}{\beta}\right)$. Therefore, we can rewrite the expression as

$$\frac{\rho}{\delta} = \frac{\Gamma - k_lA}{\Gamma} \quad (73)$$

We have determined $\delta^1$ as the solution to

$$\begin{align*}
\frac{1}{\delta^1} &= \frac{1}{\rho} - \frac{k_lA}{(\Gamma - k_lA)\delta^1} \\
\frac{\rho}{\delta^1} &= 1 - \frac{pk_lA}{(\Gamma - k_lA)\delta^1}
\end{align*}$$

We can substitute $\frac{\rho}{\delta^1}$ with the solution found above and verify that $\delta^1$ is the same

$$\frac{\Gamma - k_lA}{\Gamma} = 1 - \frac{pk_lA}{(\Gamma - k_lA)\delta^1}$$

that implies that $\delta^1$ is unique. Therefore the existence of this equilibrium requires $\delta < \delta^1$, where $\delta^1$ can be found independently as the upper threshold of the equilibrium with $\psi < 1$ or as the lower threshold of the equilibrium with $\psi = 1$. We require also that $\delta > \delta^1$, where $\delta^1$ was previously defined as

$$\delta^1 \equiv \min\left\{ \frac{m}{(1 - \rho - \varepsilon)q^* + \left(1 + \frac{2}{\beta}\frac{\varepsilon}{\rho}\right)m - k_lA}, \frac{m}{(1 - \rho + \varepsilon)q^* + \left(1 - \frac{2}{\beta}\frac{\varepsilon}{\rho}\right)m - k_lA} \right\}$$

It can be also interesting to define the value of $A$ for which the economy reach the equilibrium interest rate $R = 1$ only at the limit, i.e. the lower $A$ for which there is no possibility to have a liquidity trap equilibrium. Let define this threshold as $\tilde{A}$. We can derive it assuming $\delta^1 = 1$ and therefore

$$\begin{align*}
\frac{1}{\delta^1} &= \frac{1}{\rho} - \frac{k_lA}{m} \\
1 &= \frac{1}{\rho} - \frac{k_lA}{m} \\
1 - \rho &= \frac{k_lA}{m} \\
\frac{1 - \rho}{\rho} &= m = k_lA
\end{align*}$$

Since we know that in equilibrium

$$\frac{\gamma}{\beta} = u'\left(\frac{m}{\rho}\right)$$
and using the previous definition of \( \Gamma \equiv (u')^{-1}\left(\frac{\text{\(\hat{A}\)}}{\text{\(\hat{A}\)}}\right) \) we have

\[
\begin{align*}
    k_l \hat{A} &= \frac{1 - \rho_m}{\rho} \\
    k_l \hat{A} &= (1 - \rho) \Gamma \\
    \hat{A} &= \frac{(1 - \rho) \Gamma}{k_l}
\end{align*}
\]

(75)

that is our threshold.

B.5 Proofs

Proof of lemma 4 (Indeterminacy of \( \hat{a}_1 \) and \( \hat{b}_1 \)).

We can rewrite the first order conditions (51) and (52) as:

\[
\begin{align*}
    u'(q^{21}) &= 1 + \mu_1 - \omega_1 + \zeta_1 \\
    u'(q^{21}) &= 1 + \frac{\xi_1 - \theta_1}{k_l} + \zeta_1
\end{align*}
\]

Subtracting the second from the first we obtain:

\[
\mu_1 + \frac{\theta_1}{k_l} = \omega_1 + \frac{\xi_1}{k_l}
\]

Note that \( \mu_1 \) (\( \xi_1 \)) and \( \omega_1 \) (\( \theta_1 \)) cannot be both strictly greater than zero, because otherwise we would have both \( \hat{b}_1 = 0 \) (\( \hat{a}_1 = 0 \)) and \( b = \hat{b}_1 > 0 \) (\( a = \hat{a}_1 > 0 \)). Suppose that \( \mu_1 = 0 \) and \( \omega_1 > 0 \), that is \( b = \hat{b}_1 > 0 \). Then it must be that \( \xi_1 = 0 \) and \( \theta_1 > 0 \), so \( a = \hat{a}_1 > 0 \). Suppose instead that \( \mu_1 > 0 \) and \( \omega_1 = 0 \), that is \( b > \hat{b}_1 = 0 \). Then it must be that \( \xi_1 > 0 \) and \( \theta_1 = 0 \), so \( a > \hat{a}_1 = 0 \) (note that the converse is also true).

Consider finally the case when \( \mu_1 = 0 \) and \( \omega_1 = 0 \), that is \( b > \hat{b}_1 > 0 \), then it must be the case that also \( \xi_1 = 0 \) and \( \theta_1 = 0 \), implying that \( a > \hat{a}_1 > 0 \). In this case both (51) and (52) are equal to \( u'(q^{21}) = 1 + \zeta_1 \), giving rise to the indeterminacy of \( \hat{a}_1 \) and \( \hat{b}_1 \).

Note that we can replicate exactly the same argument with (55) and (56), obtaining therefore the same indeterminacy result for \( \hat{a}_2 \) and \( \hat{b}_2 \). □

Proof of proposition 3 (Equilibrium prices).

We start by proving the second part of the lemma. If an interbank market exists we have \( \nu_1 = \nu_2 = 0 \). Combining (57) with (54) obtains: \( u'(q^{22}) = u'(q^{12})/R \). Putting together instead (53) with (51) we get \( u'(q^{11}) = R [u'(q^{21}) - \mu_1 + \omega_1] \).

Since we are assuming the existence of the interbank market, then by lemma 4 we know that both asset and bonds will be used, therefore \( \mu_1 = 0 \). We want to show that \( \omega_1 = 0 \). Suppose \( \omega_1 > 0 \) and \( \lambda_1 > 0 \). Then from lemma 4 we would have \( \theta_1 > 0 \) and \( q^{21} = 0 \), implying \( u'(q^{21}) = \infty \) and so this cannot be a solution. Suppose instead that \( \omega_1 > 0 \) and \( \lambda_1 = 0 \), from (50) we have \( u'(q^{11}) = u'(q^{21}) \). Using it in the equation found above we have \( u'(q^{11}) = R [u'(q^{11}) + \omega_1] \), which rearranged gives \( R\omega_1 = (1 - R) R \omega_1 \), implying \( R < 1 \). However when \( R < 1 \) from (57) we have that \( (1 - R) u'(q^{22}) + \lambda_2 = \nu_2 \) which implies \( \nu_2 > 0 \) and therefore that there is no interbank market, which is a contradiction.

52
Therefore \( \mu^1 = 0 \) and \( \omega^1 = 0 \), and we can rewrite the previous equation as \( u'(q^{11}) = R \left[ u'(q^{21}) \right] \).

Using this expression and the one found above \( u'(q^{22}) = u'(q^{12})/R \) in the first order condition for the asset in the centralized market (60) we obtain

\[
R\gamma \beta = 1 + \frac{1}{2} \left[ u'(q^{11}) \right] + \frac{1}{2} \left[ u'(q^{12}) \right].
\] (76)

Substituting (50) and (54) in the first order condition for money in the centralized market (58) we can rewrite it as:

\[
\frac{\gamma}{\beta} = 1 + \frac{1}{2} \left[ u'(q^{11}) \right] + \frac{1}{2} \left[ u'(q^{12}) \right].
\] (77)

Therefore from (76) and (77) necessarily \( R = \frac{1}{\psi} \).

We can now prove the first part of the lemma. From equation (59), the first order condition with respect to bonds in the centralized market, we can see that, since \( u'(q^{21}) \geq 1, \) \( u'(q^{22}) \geq 1, \) \( \omega^1 \geq 0 \) and \( \omega^2 \geq 0 \), the RHS has to be at least equal to 1. Therefore \( \gamma \beta \geq 1 \) or equivalently \( \psi \geq \frac{2}{\beta} \).

We want to show that \( \psi \leq 1 \). Suppose there is an interbank market, then by the argument in the second part of the lemma we know that \( \omega^1 = \omega^2 = 0 \). Comparing equation (58) and (59) we can see that since \( \lambda^1 \geq 0 \) and \( \lambda^2 \geq 0 \) then \( \psi \leq 1 \).

Suppose now the interbank market does not exist, and consider the case \( \omega^1 > 0 \). Then it must be \( \mu^1 = 0 \) and from (51) it has to be \( \zeta^1 > 0 \). Therefore \( \hat{b}^1 = b \) and since the borrowing constraint is binding \( l > 0 \), but this is a contradiction since we assumed the interbank market does not exist. Hence \( \omega^1 = 0 \).

From (55) we see instead, that since by lemma \( 4 \) \( \mu^2 \) and \( \omega^2 \) cannot be both strictly greater than zero, then the solution will always imply \( \mu^2 \geq 0 \) and \( \omega^2 = 0 \). Therefore \( \omega^1 = \omega^2 = 0 \) and using the same argument as before when we assumed that there is no interbank market, we have \( \psi \leq 1 \).

We now turn to the upper bound on the asset price \( p \). By lemma \( 4 \) \( \omega^1 = \omega^2 = 0 \) implies \( \theta^1 = \theta^2 = 0 \). By combining (59) and (60) we obtain \( p = \beta \hat{k} + \beta k_1 \left( \psi - 1 \right) \), and since (for \( \gamma \geq \beta \)) \( \frac{\beta}{\gamma} \leq \psi \leq 1 \), we have that \( \beta \hat{k} \leq p \leq \beta \hat{k} + \beta k_1 \left( \frac{\gamma}{\beta} - 1 \right) \).

Proof of proposition 5. In the derivation of the equilibria section of the appendix we show that this equilibrium exists when, for a given \( A \), the following condition holds

\[
\frac{1}{\delta} \leq \frac{1}{\mu + \epsilon} - \frac{k_1 A}{m}
\] (78)

where \( m \) is the solution to

\[
\frac{\gamma}{\beta} = u' \left( \frac{m}{\delta} + k_1 A \right)
\] (79)

The RHS of (78) is decreasing in \( A \), because from (79) also \( m \) is decreasing in \( A \). Therefore also \( \delta \) should increase in order to satisfy (78), but \( \delta \) has upper bound 1. Therefore, we have to look for a threshold \( A \) such that the equilibrium exists only for \( \delta = 1 \). This implies that we need \( A \) such that \( m > 0 \) from

\[
\frac{\gamma}{\beta} = u' \left( m + k_1 A \right)
\]
\[ 1 = \frac{1}{\rho + \varepsilon} - \frac{k_l A}{m} \]

When \( A \) decreases \( m \) increases, therefore, given the continuity of \( u(\cdot) \), it exists only one \( A > 0 \) such that the system is satisfied. Once we find \( A \), it can be seen from (78) that when \( A \) decreases, \( m \) increases and a lower \( \bar{\delta}^2 \) is sufficient to satisfy (78)

\[ \frac{1}{\bar{\delta}^2} = \frac{1}{\rho + \varepsilon} - \frac{k_l A}{m} \]

When \( A = 0 \) we have \( \bar{\delta}^2 = \rho + \varepsilon \), as in the case with only government bonds.

**Proof of proposition 6.** We first show that all the thresholds are increasing in \( A \). It can be easily shown that \( \bar{\delta}^1 \) is increasing in \( A \), since \( A \) enters with a negative sign in the denominator of the definition of \( \bar{\delta}^1 \) (equation (62)) and \( m \) is exogenously fixed by the condition \( \frac{\gamma}{\beta} = u'(\frac{m}{\rho}) \).

Consider now the threshold \( \bar{\delta}^1 \), defined by condition

\[ \frac{1}{\bar{\delta}^1} = \frac{1}{\rho} - \frac{k_l A}{m} \]

Suppose that when \( A \) increases, \( \bar{\delta}^1 \) decreases. Then from the definition of \( \bar{\delta}^1 \) it implies that \( m \) is increasing more than \( k_l A \). But this violates the condition determining \( m \), \( \frac{\gamma}{\beta} = u'(\frac{m}{\rho} + k_l A) \) since arguments of the function on the right hand side have to be constant. Therefore when \( A \) increases also \( \bar{\delta}^1 \) increases.

Consider now the threshold \( \bar{\delta}^2 \), defined by condition

\[ \frac{1}{\bar{\delta}^2} = \frac{1}{\rho + \varepsilon} - \frac{k_l A}{m} \]

and \( m \) is determined by \( \frac{\gamma}{\beta} = u'(\frac{m}{\rho} + k_l A) \). Following the same argument as before it can be shown that also \( \bar{\delta}^2 \) is increasing in \( A \).

Now, we show that the distance between \( \bar{\delta}^1 \) and \( \bar{\delta}^2 \) increases when \( A \) increases. For a given \( A \), the euler for money requires that

\[ u'(\frac{m(\bar{\delta}^1, \gamma, A)}{\bar{\delta}^1} + k_l A) = u'(\frac{m(\bar{\delta}^2, \gamma, A)}{\bar{\delta}^2} + k_l A) = \frac{\gamma}{\beta} \]

or equivalently

\[ \frac{m(\bar{\delta}^1, \gamma, A)}{\bar{\delta}^1} + k_l A = \frac{m(\bar{\delta}^2, \gamma, A)}{\bar{\delta}^2} + k_l A = (u')^{-1}\left(\frac{\gamma}{\beta}\right) = \Gamma \]

Therefore, we can rewrite

\[ m(\bar{\delta}^1, \gamma, A) = \bar{\delta}^1(\Gamma - k_l A) \]
\[ m(\bar{\delta}^2, \gamma, A) = \bar{\delta}^2(\Gamma - k_l A) \]
and substituting in the conditions that determine the thresholds we have

\[
\frac{1}{\delta_1} = \frac{1}{\rho} - \frac{k_1 A}{m(\delta_1, \gamma, A)} = \frac{1}{\rho} - \frac{k_1 A}{\delta_1 (\Gamma - k_1 A)}
\]

\[
\frac{1}{\delta_2} = \frac{1}{\rho + \varepsilon} - \frac{k_1 A}{m(\delta_2, \gamma, A)} = \frac{1}{\rho} - \frac{k_1 A}{\delta_2 (\Gamma - k_1 A)}
\]

Rearranging the previous expressions we have

\[
\delta_1 = \frac{\rho \Gamma}{\Gamma - k_1 A}, \quad \delta_2 = \frac{(\rho + \varepsilon) \Gamma}{\Gamma - k_1 A}
\]

Therefore \(\delta_2 - \delta_1 = \frac{\varepsilon \Gamma}{\Gamma - k_1 A}\), increasing in \(A\) as long as \(\varepsilon > 0\).

We finally show that also the distance between \(\tilde{\delta}_1\) and \(\delta_1\) increases with \(A\). Without loss of generality consider \(\tilde{\delta}_1 = \frac{m}{(1 - \rho - \varepsilon)q^* + (1 + \frac{\varepsilon}{\rho})m - k_1 A}\), and since in this case \(\tilde{\gamma} = u'\left(\frac{m}{\rho}\right)\), substitute \(m\) with \(\rho \Gamma\), where \(\Gamma\) is defined as before. Then we have:

\[
\tilde{\delta}_1 = \frac{\rho \Gamma}{(1 - \rho - \varepsilon)q^* + \left(1 + \frac{\varepsilon}{\rho}\right)\rho \Gamma - k_1 A}, \quad \delta_1 = \frac{\rho \Gamma}{\Gamma - k_1 A}
\]

Since by definition \(\tilde{\delta}_1 < \delta_1\), then \((1 - \rho - \varepsilon)q^* + \left(1 + \frac{\varepsilon}{\rho}\right)\rho \Gamma - k_1 A > \Gamma\). This implies that as \(A\) increases, \(\delta_1\) increases more than \(\tilde{\delta}_1\). ■