The Identification of Tax Multipliers in Structural VARs*

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Structural Vector Autoregressions (SVARs) are identified imposing parametric restrictions. Each set of restrictions corresponds to a different structural model with potentially different implications. Yet the link between assumptions on the structural model and results has not been systematically explored. This paper contributes to fill this gap in the literature. First, we show how to solve analytically the translation of reduced-form residuals into policy shocks. Second, we use the closed-form solution to derive analytically impulse response functions. This theoretical framework is applied to explore identification of tax shocks. We derive an analytical relation between the output elasticity of tax revenue (restriction) and the impact tax multiplier. We show that all identification schemes used in the literature can be recasted in terms of the elasticity. Two results emerge from our analysis. First, differences in elasticity explain differences in the estimates of the tax multiplier reported in the literature. Second, empirically plausible elasticities are consistent with positive and negative tax multipliers. This finding contradicts conjectures in the literature ruling out negative tax multipliers.

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1 Introduction

Structural Vector Autoregressions (SVARs) have become a standard tool for macroeconomists. They have been estimated, among many applications, to document the effects of technology shocks (Gali (1999)), monetary policy shocks (Bernanke and Mihov (1998); Christiano, Eichenbaum, and Evans (1999); Uhlig (2005)), and fiscal shocks (Blanchard and Perotti (2002); Mountford and Uhlig (2009)).

The SVAR methodology is based on two steps. First, a reduced-form VAR is fitted to summarize the data. Second, a structural VAR model is proposed to interpret the data. In principle there are infinitely many structural models consistent with a chosen reduced-form model. From a statistical point of view all these models are equivalent. Macroeconomists select a structural model using economic arguments. The key step in VAR methodology is the identification of shocks. In principle SVARs are identified imposing a minimum of assumptions compatible with a large class of theoretical models. In practice theoretical and empirical models do not always provide guidance to select restrictions. Hence a more informal side to the identification search has developed: researchers like results to look reasonable (Uhlig (2005)). If results obtained by the selected identification scheme match the conventional wisdom, it is considered a success; otherwise the results are called a "puzzle". As stated by Uhlig (2005): "*There is danger that the literature just gets out what has been stuck in, albeit more polished and with numbers attached*". An additional informal side of SVARs analysis is the selection of the reduced-form model. Papers investigating identical questions with similar identification restrictions might use very different statistical models. The *specification search* can also be driven by the goal of obtaining reasonable results. This informal side to the SVAR approach has developed for one reason: it is hard to understand how identification restrictions and reduced-form model specification interact. The use of numerical techniques to solve SVARs limits such understanding. Macroeconimsts need a benchmark VAR model and benchmark numerical identification restrictions, even when the uncertainty surrounding both dimensions is large. Robustness of results, or selection of alternative models, is achieved by tedious numerical checks on a limited number of dimensions.

This paper attempts to fill this gap in the literature and proposes a new analytical approach to SVAR analysis. Our starting point is that the identification of one structural shock and its effects on model variables is independent of the restrictions imposed to identify the remaining shocks. In this paper we will focus on the identification of one shock, which we label policy shock. Our methodology is based on two steps. First, we show how to solve analytically the translation of reduced-form residuals into policy shocks. Second, we use the closed-form solution to derive analytically impulse response functions to the policy shock. Impulse responses are the focus of our analysis for two reasons. First, we want to unveil how the choice of restrictions and VAR models affect the results, usually described by impulse responses. Second, reasonableness of impulse responses has been used explicitly as an informal (Leeper, Sims, Zha, Hall, and Bernanke (1996)) and formal (Uhlig (2005)) identification criterion.

To illustrate our framework we use a bi-variate model. Typical identification problems faced by macroeconomists can be reformulated in terms of a policy instrument and a non-policy variable. The objects of interest are the impact responses to a policy shock. To identify the model a restriction is imposed on the movements in the policy instrument due to movements in the non-policy variable. The restriction is selected from a (possibly unbounded) set of restrictions. This simple baseline case is sufficient for establishing three important results that generalize to multivariate models and impulse responses at longer horizons. First, impact responses to a policy shock, function of the identification restriction, are defined over a compact set. Second, we characterize two non-empty sets of impact responses of opposite sign. Third, we state properties of the relation between impulse responses and restrictions (e.g. monotonicity, concavity) that hold in any VAR model.

We apply our methodological findings to study the effects of tax shocks. This example is illustrative because the identification of tax shocks rely on one key identification restriction, imposed on the automatic response of tax revenue to economic activity. Two seminal papers (Blanchard and Perotti (2002); Mountford and Uhlig (2009)) estimate and positive and large response of output to tax cuts, while subsequent studies (e.g. Perotti (2005); Favero and Giavazzi (2009)) find that output does not react to tax cuts up to one year.

We derive an analytical relation between the output elasticity of tax revenue (restriction) and the impact tax multiplier. We show that all identification schemes used in the literature can be recasted in terms of the elasticity. Two results emerge from our analysis. First, differences in elasticity explain differences in the estimates of the tax multiplier reported in the literature. Second, empirically plausible elasticities are consistent with positive and negative tax multipliers. This finding contradicts conjectures in the literature ruling out negative tax multipliers.

2 Analytics

2.1 The Structural Model

Our basic structural VAR specification is

$$A_0 Y_t = \sum_{l=1}^{p} A_l Y_{t-l} + Cz + \epsilon_t, \text{ for } 1 \leq t \leq T, \tag{1}$$

where:

- p is the lag length,
- T is the sample size,
- $Y_t = [X_t, T_t]$ is a two dimensional vector in the logarithms of quarterly GDP and taxes, all in real, per capita, terms,

- z is an $n_z \times 1$ vector of deterministic terms,
- ϵ_t is a 2×1 vector of exogenous structural shocks,
- A_l is an 2×2 matrix of parameters for $0 \leq l \leq p$, and
- C is an $n_z \times 2$ matrix of parameters.

The initial conditions $Y_0, \dots Y_{1-p}$ are taken as given. The distribution of ϵ_t, conditional on the past information, is Gaussian with mean zero and diagonal covariance matrix Ω. We denote the standard deviation of the structural shocks by ω_Y and ω_T. In appendix A we provide details of the SVAR specification used for the empirical exercise. We estimate the deterministic specification described by Blanchard and Perotti (2002, p. 1332).

The reduced-form representation implied by the structural model (1) is

$$Y_t = \sum_{l=1}^{p} B_l Y_{t-l} + Dz + u_t$$

where $B_l = A_0^{-1} A_l$ for $1 \leq l \leq p$, $D = A_0^{-1} C$, and u_t is a 2×1 vector of reduced-form residuals with mean zero and symmetric covariance matrix Σ:

$$\Sigma = \begin{bmatrix} \sigma_{YY} & \sigma_{YT} \\ \sigma_{YT} & \sigma_{TT} \end{bmatrix} \tag{2}$$

The relation between reduced-form residuals and structural shocks is:

$$A_0 u_t = \epsilon_t \tag{3}$$

and the relation between the reduced-form coefficients Σ and the structural coefficients (A_0, Ω) is given by the second moment of u_t and ϵ_t:

$$\begin{aligned} \mathbb{E}\left[u_t u_t'\right] &= \mathbb{E}\left[\epsilon_t \epsilon_t'\right] \\ \Sigma &= A_0^{-1} \Omega A_0^{-1'} \end{aligned} \tag{4}$$

Following the notation used by Blanchard and Perotti (2002), we write system (3) as:

$$\begin{aligned} u_t^Y &= c_1 u_t^T + \epsilon_t^Y \\ u_t^T &= a_1 u_t^Y + \epsilon_t^T \end{aligned}$$

where

$$A_0 = \begin{bmatrix} 1 & -c_1 \\ -a_1 & 1 \end{bmatrix} \tag{5}$$

Unexpected movements in GDP $\left(u_t^Y\right)$ can be due to two factors: the response to unexpected movements in tax revenue, captured by $c_1 u_t^T$, and non-policy shocks ϵ_t^Y Similarly, unexpected movements in tax revenue $\left(u_t^T\right)$ can be due to the response to unexpected movements in GDP, captured by $a_1 u_t^Y$, and structural tax shocks ϵ_t^T. Similarly unexpected movements in GDP $\left(u_t^Y\right)$ can be due to two factors: the response to unexpected movements in tax revenue, captured by $c_1 u_t^T$, and to non-policy shocks ϵ_t^Y.

We can use a simple Real Business Cycle (RBC) model to interpret the structural shocks. ϵ_t^T represents a shock to a proportional tax rate levied on labor income, capital income and consumption. ϵ_t^Y can represent a shock to technology.

The impact response of GDP and tax revenue (expressed in deviations from z) to structural shocks is:

$$\begin{aligned} u_t &= A_0^{-1}\epsilon_t \\ \begin{bmatrix} u_t^Y \\ u_t^T \end{bmatrix} &= \frac{1}{1-a_1c_1}\begin{bmatrix} 1 & c_1 \\ a_1 & 1 \end{bmatrix}\begin{bmatrix} \epsilon_t^Y \\ \epsilon_t^T \end{bmatrix} \end{aligned}$$

The first column of matrix A_0^{-1} is the impulse vector associated to a tax shock: the first row denotes the response of tax revenue to a tax shock and the second row the response of GDP. Similarly, the second column of denotes the impulse vector associated to a non-policy shock.

To identify a bi-variate SVAR we need to impose one restriction, either on c_1 or a_1. The coefficient c_1 captures the response of output to a 1% increase in tax revenue:

$$c_1 = \frac{\partial u_t^Y / \epsilon_t^T}{\partial u_t^T / \epsilon_t^T} \tag{6}$$

Theoretical models predict that $c_1 < 0$: an tax exogenous shock ϵ_t^T that increases tax revenue by 1% provokes a decline in output. Restrictions on this coefficient are not imposed for two reasons. First, theoretical models do not have sharp predictions about the size of c_1. Second, estimates of c_1 can be used to validate and calibrate theoretical models.

The coefficient a_1 denotes the output elasticity of tax revenue:

$$a_1 = \frac{\partial u_t^T / \epsilon_t^Y}{\partial u_t^Y / \epsilon_t^Y} \tag{7}$$

In public finance literature this elasticity serves as an indicator of the overall progressivity of the tax system. A proportional income tax has an elasticity of 1.0, while progressive tax systems whose tax-income ratios increase with income have an elasticity greater than 1.0. This elasticity also serves as an indicator of the tax system's role as an automatic stabilizer. Assuming that $c_1 < 0$ and $a_1 > 0$, for a given level of taxes, the higher the elasticity, the smaller will be the change in GDP $\left(u_t^Y\right)$ that results from an exogenous shock $\left(\epsilon_t^Y\right)$. Several international organizations and national governments provide figures for this elasticity, either constructing it from statutory tax rates, or estimating it. Hence SVAR models have been identified imposing restrictions on this elasticity.

2.2 Analytical Solution

Equation (4) denotes a system of three non-linear equations (as many as the distinct elements of Σ) in four unknowns $(a_1, c_1, \omega_Y, \omega_T)$. As explained in the previous paragraph, we assume that a_1 is restricted, i.e. we have off-model information about this structural coefficient. The system becomes exactly identified and a unique solution exists. In the SVAR literature system (4) is solved numerically. Instead we solve system (4) analytically. We find four candidate triples $(c_1, \omega_Y, \omega_T)$. Three candidate solutions imply negative standard deviations for the structural shocks (ω_Y, ω_T) for any Σ. The analytical derivation and the complete solution is reported in appendix B.

In the interest of brevity we only discuss the solution for c_1:

$$c_1\left(a_1;\Sigma\right) = \frac{\sigma_{YT} - a_1\sigma_{YY}}{\sigma_{TT} - a_1\sigma_{YT}} \tag{8}$$

This analytical expression provides a non-linear mapping between the output elasticity of tax revenue a_1 and the tax revenue elasticity of output c_1. The mapping depends on the elements of the variance-covariance matrix Σ of reduced-form residuals. Figure 1 plots $-c_1\left(a_1;\hat{\Sigma}_{OLS}\right)$, where $\hat{\Sigma}_{OLS}$ denotes the OLS estimate obtained estimating the VAR described in appendix A. $-c_1$ indicates the response of output to a tax shock that decreases tax revenue by 1%. The function $c_1\left(a_1;\Sigma\right)$ has two interesting properties that hold for any Σ. First, c_1 is zero if and only if $a_1 \equiv \overline{a}_1 = \sigma_{YT}/\sigma_{YY}$. If $c_1 = 0$, the impact response of model variables to structural shocks become:

$$\begin{bmatrix} u_t^Y \\ u_t^T \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \sigma_{YT}/\sigma_{YY} & 1 \end{bmatrix} \begin{bmatrix} \epsilon_t^Y \\ \epsilon_t^T \end{bmatrix}$$

To impose $a_1 = \overline{a}_1$ is equivalent to identify the SVAR taking a Cholesky factorization of matrix Σ and assuming that GDP is ordered before tax revenue (Sims, 1980). If instead we impose $a_1 = 0$, we get $c_1 = \sigma_{YT}/\sigma_{TT}$ and

$$\begin{bmatrix} u_t^Y \\ u_t^T \end{bmatrix} = \begin{bmatrix} 1 & \sigma_{YT}/\sigma_{TT} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_t^Y \\ \epsilon_t^T \end{bmatrix}$$

To impose $a_1 = 0$ is equivalent to identify the SVAR taking a Cholesky factorization of matrix Σ and assuming that tax revenue is ordered before GDP. Notice that if $a_1 < \overline{a}_1$ a 1% decline in tax revenue is associated to a *decline* in GDP. We can gain intuition examining the case $a_1 = 0$. A robust finding in the literature is that $\sigma_{YT} > 0$. When $a_1 = 0$ the tax shock need to explain the entire sample covariance σ_{YT}, and hence a decline taxes must be associated to a decline in GDP.

Second:

$$\begin{aligned} \lim_{a_1 \to a_{1,lim}^-} c_1\left(a_1;\Sigma\right) &= +\infty \\ \lim_{a_1 \to a_{1,lim}^+} c_1\left(a_1;\Sigma\right) &= -\infty \end{aligned}$$

where $a_{1,lim} = \sigma_{TT}/\sigma_{YT}$. Notice that $a_{1,lim} > \overline{a}_1$ for any Σ such that $|\rho_{YT}| < 1$, where ρ_{YT} is the correlation coefficient corr $\left(u_t^Y, u_t^T\right)$. The restriction $a_{1,lim}$ does note define a SVAR model. This restriction will be relevant when we analyze sign restrictions in section 4.

Some authors use $-c_1$ as measure of the impact tax multiplier (e.g. Mountford and Uhlig, 2009). According to this definition, even if we exclude values of a_1 around the discontinuity and larger than $a_{1,lim}$, the tax multiplier can be potentially very large. This measure of the effectiveness of tax policy suffers from an important drawback: it does not consider the macroeconomic feedback in the economy, i.e. the presence of automatic stabilizers.

To better understand the role of automatic stabilization, define the response of tax revenue to a one dollar tax shock:

$$
\begin{aligned}
TR_0\left(a_1;\Sigma\right) &= (-1)\frac{1}{1-a_1c_1\left(a_1;\Sigma\right)} \\
TR_0\left(a_1;\Sigma\right) &= \frac{a_1\sigma_{YT}-\sigma_{TT}}{a_1^2\sigma_{YY}-2a_1\sigma_{YT}+\sigma_{TT}}
\end{aligned}
$$

At $a_1 = 0$ the response of tax revenue $\left(u_t^T\right)$ to a tax shock $\left(\epsilon_t^T = -1\right)$ is -1. Inspecting equation () we see that ax revenue does not react to movements in GDP. When $a_1 = \overline{a}_1$ the response of tax revenue is also -1. The reason is that GDP does not react to movements in tax revenue. For $\overline{a}_1 < a_1 < a_{1,lim}$ the response of tax revenue increases in a_1. The larger a_1, the more GDP reacts to a given change in tax revenue (c_1), the smaller the decline in tax revenue. The intuition s that a tax cut increases the tax base. Hence the size of a tax shock ϵ_t^T necessary to generate a decline in tax revenue of size 1% is increasing in a_1, as can be seen in Figure 4. For $a_1 > a_{1,lim}$ a negative tax shock generates an increase in tax revenue. This is due to the fact that in this interval for a_1 GDP declines after a tax cut.

Tax Multiplier. Following Blanchard and Perotti, 2002 we define the tax multiplier as the dollar response of output to a tax shock of size one dollar:

$$
\begin{aligned}
TM_0\left(a_1;\Sigma\right) &= (-1)\quad\frac{c_1\left(a_1;\Sigma\right)}{1-a_1c_1\left(a_1;\Sigma\right)}\frac{1}{\overline{\overline{T}/\overline{Y}}} \\
TM_0\left(a_1;\Sigma\right) &= \frac{a_1\sigma_{YY}-\sigma_{TY}}{a_1^2\sigma_{YY}-2a_1\sigma_{YT}+\sigma_{TT}}\frac{1}{\overline{\overline{T}/\overline{Y}}}
\end{aligned}
$$

where -1 denotes the size of the shock, and $\overline{T}/\overline{Y}$ denotes the mean tax-to-GDP ratio. This scaling factor transforms percentage changes into dollar changes. The following theorem states some key properties of the impact tax multiplier $TM_0\left(a_1;\Sigma\right)$.

Proposition 1 *For any positive-definite variance-covariance matrix Σ, the impact tax multiplier $TM_0\left(a_1;\Sigma\right)$ has the following properties:*

1. *It has a unique global minimum:*

$$
TM_0\left(a_1^{min};\Sigma\right) = -\frac{\sigma_Y}{\sigma_T}\left(\frac{1}{2\left(1-\rho_{TY}^2\right)^{0.5}}\right)\frac{1}{\overline{\overline{T}/\overline{Y}}} < 0
$$

where $a_1^{min} = (\sigma_T/\sigma_Y)\left(\sqrt{1-\rho_{TY}^2} - \rho_{TY}\right)$.

2. *It has a unique global maximum:*

$$TM_0\left(a_1^{max};\Sigma\right) = \frac{\sigma_Y}{\sigma_T}\left(\frac{1}{2\left(1-\rho_{TY}^2\right)^{0.5}}\right)\frac{1}{\overline{T}/\overline{Y}} > 0$$

where $a_1^{max} = (\sigma_T/\sigma_Y)\left(\sqrt{1-\rho_{TY}^2} + \rho_{TY}\right)$.

3. *It has a unique zero:*

$$TM_0\left(\overline{a}_1;\Sigma\right) = 0 \Longleftrightarrow \overline{a}_1 = \frac{\sigma_{TY}}{\sigma_{YY}}$$

4. *It is such that:*

$$\lim_{a_1 \to \pm\infty} TM_0\left(a_1;\Sigma\right) = 0 \tag{9}$$

Proof The proof is provided in Appendix A. ||

Figure 4 plots $TM_0\left(a_1;\hat{\Sigma}_{OLS}\right)$. The impact multiplier is a bounded function of a_1. The set of admissible values of the impact multiplier ranges between -1.05 and 1.05. The lower bound is associated to a value of the elasticity $a_1^{min} = -1.62$, while the upper bound is associated to a value of the elasticity $a_1^{max} = 3.84$. Since the bounds have opposite sign and $TM_0\left(a_1;\Sigma\right)$ is a continuous function in a_1, the zero multiplier is always included. Not surprisingly, the impact multiplier is zero at $a_1 = \overline{a}_1$, the elasticity that associated to $c_1 = 0$. In our numerical example $\overline{a}_1 = 1.11$. Values of $a_1 < \overline{a}_1$ are associated to negative impact multipliers, as it is the case for $-c_1\left(a_1;\Sigma\right)$. Values of $a_1 > \overline{a}_1$ are associated to positive impact multipliers.

In this section we have shown how to characterize analytically the identification problem in a bi-variate SVAR model. The main message is that sign and the size of the impact tax multiplier depend on the choice of the output elasticity of tax revenue a_1 in any VAR model. This elasticity is a random variable. In the next three sections we discuss how to compute distributions for $a_1$using three different approaches.

3 Elasticity from OECD estimates.

This section builds on the seminal work of Blanchard and Perotti, 2002. These authors estimate the output elasticity of tax revenue following the methodology adopted by the OECD and described in Giorno, Richardson, Roseveare, and Van den Noord, 1995.

Blanchard and Perotti, 2002 restrict a_1 to the point estimate, which equals to 2.16 for the period $1960:1-1997:4$. As we have argued in the previous section, the impact tax multiplier varies substantially with respect to a_1. The goals of this section are to provide a measure of uncertainty around the point estimate provided by Blanchard and Perotti, 2002, and to assess how uncertainty around a_1 translates into uncertainty around the impact tax multiplier.

In the current draft of the paper we make an arbitrary choice and we assume that $a_1 \sim Beta(2, 2, 1, 3)$ where the first two parameters control the shape of the distribution while the last two parameters define the support of the distribution. The mean and median of a_1 is 2, a value close to the to choice of Blanchard and Perotti, 2002. We believe that is reasonable to assume that a 1% increase increase in output leads to an increase in tax revenue between 1 and 3%. Values of a_1 outside this interval seem extremely unlikely. Figure 5 plots the distribution of a_1. we can interpret the identification strategy followed by Blanchard and Perotti, 2002 as assigning probability 1 to $a_1 = 2.16$.

Figure 6 plots the posterior distribution of the impact tax multiplier, conditional on $\Sigma = \hat{\Sigma}_{OLS}$, i.e. we abstract from sampling uncertainty. The median multiplier is 0.62, a value which is very close to the point estimate of Blanchard and Perotti, 2002. The 95% credible set includes values of the multiplier ranging between 0.06 and 0.95, a value close to the maximum multiplier of 1.05.

Since uncertainty around a_1 translates into large uncertainty around the impact tax multiplier, in the next version of the paper we will provide a measure of uncertainty based on the OECD methodology. In particular we will use the new methodology adopted by the OECD to measure the elasticities and discussed in Girouard and André, 2005. These authors estimate the output elasticity of tax revenue a_1 for the United States and other 23 developed economies. Four different types of taxes are distinguished: personal income tax; social security contributions corporate income tax and indirect taxes. We denote by $\eta_{T_i,x}$ the elasticity of the $i-th$ tax category with respect to the output gap. The elasticity $\eta_{T_i,x}$ can be separated in two components, an elasticity of tax revenue with respect to the relevant tax base, η_{T_i,TB_i}, and an elasticity of the tax base relative to the cyclically adjusted indicator, $\eta_{TB_i,x}$:

$$\eta_{T_i,x} = \eta_{T_i,TB_i}\eta_{TB_i,x} \tag{10}$$

The elasticities of taxes with respect to their base are extracted from tax legislation and related fiscal data, while the sensitivity of the different tax bases with respect to the output gap is estimated using time-series data. The overall output elasticity of tax revenue is:

$$a_1 = \sum_i \eta_{T_i,x} \frac{T_i}{T}$$

where i indexes the tax category. We review how elasticities are constructed. We argue that measurement errors in the constructed η_{T_i,TB_i} and standard errors in $\eta_{TB_i,x}$ should be taken into account.

3.1 Elasticities of tax receipts and expenditures with respect to their base

3.1.1 Elasticities of personal income and social security contributions based on tax rules and revenue data.

To calculate the elasticity of income tax (social security contributions) with respect to the tax base, the marginal and average tax rates of a representative household are first calculated for several points in the earning distribution. The weighted averages of the marginal and average tax rates are then computed. The weights of the various earning levels are derived from an estimated earning distribution. Formally, per capita elasticity of income tax (social security contribution) with respect to earnings is expressed as follows:

$$\eta_{T,W} = \left(\sum_{i=1}^{n} \gamma_i MA_i \right) / \left(\sum_{i=1}^{n} \gamma_i AV_i \right) \tag{11}$$

where γ_i is the weight of earnings-level i in total earnings expressed in currency units earned (the first-moment distribution), MA_i is the marginal income tax rate (social contribution rate) at point i on the earnings distribution, and AV_i is the average income tax rate (social contribution rate) at point i on the earnings distribution. This elasticity is then applied to the cyclical variation in the aggregate wage bill.

Main message: measurement errors are likely and potentially large. Need to investigate further.

3.1.2 Corporate income tax and indirect tax.

Corporate income tax revenue are assumed to be proportional to the tax base (profits). Similarly indirect taxes are taken to be proportional to their main tax base, consumer expenditure.

3.2 Elasticities of tax and expenditure bases with respect to cyclical indicators.

3.2.1 Cyclical sensitivity of the income tax and social security.

The sensitivity of the income tax and social security contributions tax bases with respect to the cycle has been estimated using the following equation:

$$\triangle \log \left(W_t L_t / X_t^* \right) = \alpha_0 + \alpha_1 \triangle \log \left(X_t / X_t^* \right) \tag{12}$$

where X_t^* is potential output, W is the wage rate and L is employment. The coefficient α_1 is the elasticity of the wage bill relative to the output gap, $\eta_{WL,x}$.

Main message: we need to compute the standard error of α_1.

3.2.2 Cyclical sensitivity of the indirect tax base.

This elasticity is set to unity. Girouard and Andre' (2005) attempt to estimate this elasticity using an equation linking real private consumption to output gap. They find wide standard errors associated to the regression coefficients and argue that there might be severe endogeneity problems.

Main message: we can add a measurement error. Need to investigate further.

3.2.3 Cyclical sensitivity of the corporate tax base.

The elasticity of profits with respect to the output gap can be derived from the elasticity of the wage bill with respect to the output gap:

$$\eta_{P,x} = \frac{(1\,(1-PS)\,\eta_{WL,x})}{PS}$$

where PS is the profit share in GDP. Blanchard and Perotti, 2002 instead measure $\eta_{P,x}$ regressing changes in profits on changes in output. Back of the envelope calculations suggest that the two approaches give different results.

4 Sign Restrictions

The sign restriction approach has been used by Mountford and Uhlig (2009) to identify fiscal policy shocks. These authors identify two shocks. First, a business cycle shock is identified to capture movements in model variables not attributable to fiscal policy. In our small model this shock is ϵ_t^Y, the GDP shock. We assume that GDP and tax revenue increase on impact following a business cycle shock. Second, they identify a tax shock. We assume that tax revenue increases on impact following a tax shock[1]. The response of output to a tax shock is left unrestricted. The impact response of model variables to structural shocks:

$$\begin{aligned}
A_0^{-1} &= \frac{1}{1-a_1c_1}\begin{bmatrix} 1 & c_1 \\ a_1 & 1 \end{bmatrix} \\
A_0^{-1}(a_1;\Sigma) &= \frac{1}{\det(A_0)}\begin{bmatrix} \sigma_{TT}-a_1\sigma_{TY}, & \sigma_{TY}-a_1\sigma_{YY} \\ a_1(\sigma_{TT}-a_1\sigma_{TY}), & \sigma_{TT}-a_1\sigma_{TY} \end{bmatrix} \\
&= \begin{bmatrix} + & ? \\ + & + \end{bmatrix}
\end{aligned}$$

where $\det(A_0) = a_1^2\sigma_{YY} + \sigma_{TT} - 2a_1\sigma_{TY}$. The first column of matrix A_0^{-1} is the impulse vector associated to a tax shock: the first row denotes the response of tax revenue to a tax shock and the second row the response of GDP. Similarly, the second column of denotes the impulse vector associated to a GDP shock.

[1]Mountford and Uhlig (2009) identify the business cycle shock imposing additional restrictions on consumption and investment. Furthermore restrictions are imposed up to four quarters after the shocks.

Sign restrictions are satisfied if and only if

$$0 < a_1 < a_1^{lim} \tag{13}$$

where $a_{1,lim} = \sigma_{TT}/\sigma_{YT}$. Recall from section 2 that the tax multiplier is zero when $a_1 = \overline{a}_1$, where σ_{TY}/σ_{YY}. For any variance covariance matrix Σ such that $\sigma_{YT} > 0$ we know that:

$$0 < \overline{a}_1 < a_{1,lim}$$

Since $\overline{a}_1$ is strictly included in the set of sign restriction solutions, the sign of the multiplier cannot be determined. That is, there always exists an arbitrarily small δ such that

$$\begin{array}{rcl} TM_0\left(a_1 = \overline{a}_1 - \delta; \Sigma\right) & < & 0 \\ TM_0\left(a_1 = \overline{a}_1 + \delta; \Sigma\right) & > & 0 \end{array}$$

and sign restrictions are satisfied.

We have shown that sign restrictions can be mapped into an interval for the output elasticity of tax revenue a_1. In the remaining of the section we show how a_1 is distributed over the support $(0, a_{1,lim})$ following the implementation of the sign restriction approach proposed by Uhlig (2005). Define by $\tilde{A}_0^{-1}$ the lower triangular Cholesky factor of matrix Σ:

$$\begin{array}{rcl} \tilde{A}_0^{-1} & = & A_0^{-1}\left(a_1 = \overline{a}_1; \Sigma\right)\Omega^{1/2}\left(a_1 = \overline{a}_1; \Sigma\right) \\ & = & \left[\begin{array}{cc} 1 & 0 \\ \sigma_{YT}/\sigma_{YY} & 1 \end{array}\right]\left[\begin{array}{cc} \sigma_Y & 0 \\ 0 & \sigma_T\left(1 - \rho_{YT}^2\right)^{0.5} \end{array}\right] \\ & = & \left[\begin{array}{cc} \sigma_Y & 0 \\ \sigma_T\rho_{YT} & \sigma_T\sqrt{1 - \rho_{YT}^2} \end{array}\right] \end{array}$$

Define the rotation matrix:

$$Q = \left[\begin{array}{cc} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{array}\right]$$

where θ is the rotation angle and $QQ' = I$, $\det(Q) = I$. Finally define:

$$\left[\begin{array}{cc} F_{11} & F_{12} \\ F_{21} & F_{22} \end{array}\right] = \left[\begin{array}{cc} \sigma_1 & 0 \\ \sigma_2\rho_{12} & \sigma_2\sqrt{1 - \rho_{12}^2} \end{array}\right]\left[\begin{array}{cc} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{array}\right], \tag{14}$$

Notice that $\Sigma = FF'$ for any rotation angle $\theta \in [-\pi, \pi]$. Let $\varphi \equiv \arccos(\rho_{YT})$ be the angle representation of the error correlation coefficient. Using this definition we can express $\rho_{YT} = \cos(\varphi)$ and $\sqrt{1 - \rho_{YT}^2} = \sin(\varphi)$.[2] Using the angle

[2] Note that $\rho_{YT} \in (-1, 1)$ implies $\varphi \in (0, \pi)$. The angle φ is strictly decreasing in ρ_{YT}, with $\varphi \to \pi$ as $\rho_{YT} \to -1$, $\varphi = \frac{\pi}{2}$ for $\rho_{YT} = 0$ and $\varphi \to 0$ as $\rho_{YT} \to 1$.

representation of ρ_{YT}, the expressions for the impact impulse responses implied by Equation (14) simplify to (with sign restrictions shown where applicable):

$$
\begin{aligned}
F_{11} &= \sigma_Y \cos(\theta) > 0, && (15)\\
F_{21} &= \sigma_T \cos(\theta - \varphi) > 0, && (16)\\
F_{12} &= -\sigma_Y \sin(\theta), && (17)\\
F_{22} &= -\sigma_T \sin(\theta - \varphi) > 0, && (18)
\end{aligned}
$$

where we have made use of basic trigonometric identities.[3]. We can now analytically characterize the set of all pure sign restriction solutions.

Proposition 2 *Let S be the set of all solutions satisfying the sign restrictions. Then, the set S, for given $\varphi \in (0, \pi)$, is*

$$S \equiv \left\{ \theta \in [-\pi, \pi] : -\frac{\pi}{2} + \varphi < \theta < \min(\varphi, \frac{\pi}{2}) \right\}.$$

The output elasticity of tax revenue a_1 is defined as the ratio between the tax response and the output response to a business cycle shock. Hence we can express a_1 as:

$$
\begin{aligned}
a_1(\theta; \Sigma) &= \frac{F_{21}(\theta)}{F_{11}(\theta)}\\
&= \frac{\sigma_T \cos(\theta - \varphi)}{\sigma_Y \cos(\theta)} \qquad (19)
\end{aligned}
$$

where $\theta \in S$. Equation (19) maps rotation angles, which do not have an economic interpretation, into output elasticity of tax revenue. Uhlig (2005) assumes that θ is distributed uniformly over $[-\pi, \pi]$. Figure 6 plots the distribution of a_1 conditional on $\Sigma = \hat{\Sigma}_{OLS}$. The median elasticity is 2.26, a value close to the elasticity used by Blanchard and Perotti (2002). The distribution of a_1 however puts probability mass over extremely low and high values of a_1, which do not seem to be plausible. Table 1 reports the 95% credible set for a_1, which includes values between 0.13 and 7.11.

Figure 7 shows that posterior distribution of the impact tax multiplier conditional on $\Sigma = \hat{\Sigma}_{OLS}$. The median multiplier is 0.75, and not surprisingly uncertainty is very large: the 95% credible set includes values for the multiplier between -0.67 and 1.05.

Penalty Function Approach

Mountford and Uhlig (2009) do not report results for the pure sign restriction approach. Instead they select an element form the set of sign restriction solutions S using a criterion function. This approach is known as the "penalty function" approach. The penalty function in our simple bi-variate model can be written as:

$$\max_{\theta} \frac{F_{11}(\theta)}{\sigma_Y} + \frac{F_{21}(\theta)}{\sigma_T} = \cos(\theta) + \cos(\theta - \varphi) \qquad (20)$$

[3]The expression for F_{21} uses the angle difference identity $\cos(\theta)\cos(\varphi) + \sin(\theta)\sin(\varphi) = \cos(\theta - \varphi)$, while the expression for F_{22} uses the angle difference identity $\sin(\theta)\cos(\varphi) - \cos(\theta)\sin(\varphi) = \sin(\theta - \varphi)$.

subject to sign restrictions. Caldara and Kamps (2010) show that the inequality constraints associated with the sign restrictions are automatically satisfied. This penalty function selects the rotation angle θ that maximizes the forecast error variance of GDP and tax revenue explained by ϵ_t^Y. The analytical solution to this problem, assuming $\rho_{TY} > 0$ is

$$\theta^*(\varphi) = \arctan\left(\frac{\sin\varphi}{1+\cos\varphi}\right)$$

Table 1 reports the output elasticity of tax revenue implied by $\theta^*(\hat{\varphi})$, which is equal to 3.28. This value is substantially larger than the elasticity used by Blanchard and Perotti (2002) and by the median elasticity implied by the pure sign restriction approach. The associated impact tax multiplier is 1.03, a value close to the maximum admissible multiplier 1.05.

The penalty function approach can be interpreted as imposing a dogmatic prior on a_1 conditional on Σ. Differences between this choice of a_1 and the choice of a_1 operated by Blanchard and Perotti (2002) seem to explain most of the difference in results reported in the original papers. A complete assessment can be done only introducing sampling uncertainty, as we will be able to compute the posterior distribution of a_1 consistent with the penalty function approach.

5 Back of the Envelope Calculation from DSGE models.

What is the output elasticity of tax revenue in a DSGE model? How large is it?

Assume that the production sector consists of a continuum of firms operating in a competitive market. Further assume that firms produce output X_t using a Cobb-Douglas production function:

$$X_t = a_t K_t^{\alpha} L_t^{1-\alpha}$$

where a_t denotes an exogenous technology process. For simplicity assume that prices and wages are fully flexible. Cost minimization implies that capital and labor are paid a constant share of output:

$$R_t K_t = \alpha X_t \tag{21}$$

$$W_t L_t = (1-\alpha) X_t \tag{22}$$

Assume that the government taxes labor income $W_t L_t$ at a rate $\tau_{L,t}$ and capital income $R_t K_t$ at a rate $\tau_{L,t}$. Capital income is taxed after an allowance for depreciation. The tax revenue Tax_t is equal to:

$$Tax_t = +\tau_{K,t} R_t K_t + \tau_{L,t} W_t L_t \tag{23}$$

Taxes are used to finance public consumption G_t:

$$Tax_t = G_t$$

Plugging in (21) and (22) into (23) and log-linearizing we get:

$$\hat{Tax}_t = \frac{\tau_K \alpha}{\tau_K \alpha + \tau_L (1-\alpha)} \left[\hat{X}_t + \hat{\tau}_{K,t}\right] + \frac{\tau_L (1-\alpha)}{\tau_K \alpha + \tau_L (1-\alpha)} \left[\hat{X}_t + \hat{\tau}_{L,t}\right] \quad (24)$$

where $\hat{var}_t$ denote percentage deviations from steady state and τ_K and τ_L denote steady state tax rates.

Assume that tax rates follow an exogenous i.i.d. process. To compute the output elasticity of tax revenue assume that the economy is hit by a technology shock that increases output by 1% of his steady state value. Then tax revenue increases by

$$\begin{aligned} \hat{Tax}_t &= \frac{\tau_K \alpha}{\tau_K \alpha + \tau_L (1-\alpha)} + \frac{\tau_L (1-\alpha)}{\tau_K \alpha + \tau_L (1-\alpha)} \\ &= 1 \end{aligned}$$

Hence the output elasticity of tax revenue is one.

Let us assume that tax policy follows a simple rule, as in Leeper, Plante, and Traum (2009):

$$\hat{\tau}_{K,t} = \varphi_K \hat{X}_t + \epsilon_t^{\tau_K}$$

$$\hat{\tau}_{L,t} = \varphi_L \hat{X}_t + \epsilon_t^{\tau_L}$$

Tax rates respond to the cyclical position of the economy. This fiscal rule captures the automatic stabilization role of the tax system. What is the output elasticity of tax revenue? Plugging the fiscal rules in the expression for the tax revenue and simplifying we get:

$$\hat{Tax}_t = \frac{\tau_K \alpha}{\tau_K \alpha + \tau_L (1-\alpha)} (1+\varphi_K) + \frac{\tau_L (1-\alpha)}{\tau_K \alpha + \tau_L (1-\alpha)} (1+\varphi_L)$$

Leeper, Plante, and Traum (2009) estimates a a DSGE model of fiscal policy using Bayesian techniques. To get a rough idea of the distribution of the output elasticity of tax revenue implied by a theoretical model we follow their specification of prior distributions over structural parameters. The capital share of production α is calibrated to 0.3. The steady state capital income tax rate and labor income tax rate are set to 0.184 and 0.223 respectively. The calibration of these parameters is standard and it does not depend on the exact specification of the model. The parameter φ_K is assumed to have a gamma distribution with mean 1 and standard deviation 0.3. The parameter φ_L is assumed to have a gamma distribution with mean 0.5 and standard deviation 0.25. Figure 7 plots the prior distribution of the output elasticity of tax revenue implied by this choice of priors. The median is 1.60, and the 95% credible set ranges between 1.30 and 2.10. Notice that both the value of $a_1$used by Blanchard and Perotti (2002) and the value of a_1 consistent with the penalty function approach lie outside this credible set.

Figure 8 plots the posterior distribution of the impact tax multiplier conditional on $\Sigma = \hat{\Sigma}_{OLS}$ for the distribution of a_1 derived from the DSGE model. The median multiplier is 0.37, and the 95% credible set is $(0.16, 0.67)$. Not surprisingly the median multiplier is lower compared to the multiplier obtained from the other approaches, and the credible set is smaller.

In our discussion we have abstracted from indirect taxes, which could be represented by a proportional tax rate on consumption $\tau_{C,t}$. The government does not use indirect taxes to stabilize the economy, i.e. it is reasonable to assume that $\varphi_C = 0$. Hence the output elasticity of tax revenue from indirect taxes equals the output elasticity of consumption. This elasticity depends on the specification of the structure of the economy. In a model with a CRRA utility function this elasticity is below 1. In the next version of the paper we will provide a formal discussion about this elasticity.

6 Bayesian Estimation

[TO BE ADDED]

7 Conclusions

[TO BE ADDED]

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Method	Distribution of a_1		$p\left(TM_0\vert\hat{\Sigma}_{OLS}\right)$	
	Median	95% C.S.	Median	95% C.S.
Blanchard&Perotti (2002)	2.16	-	0.69	-
OECD + Uncertainty	2.00	(1.19, 2.81)	0.62	(0.06, 0.95)
Sign Restrictions	2.26	(0.13, 7.11)	0.75	(-0.67, 1.05)
Sign Restriction + P.F.	3.28	-	1.03	-
DSGE	1.60	(1.32, 2.10)	0.37	(0.16, 0.67)

Table 1: Distributions of the output elasticity of tax revenue implied by different methods. Posterior distributions of impact tax multipliers conditional on $\hat{\Sigma}_{OLS}$.

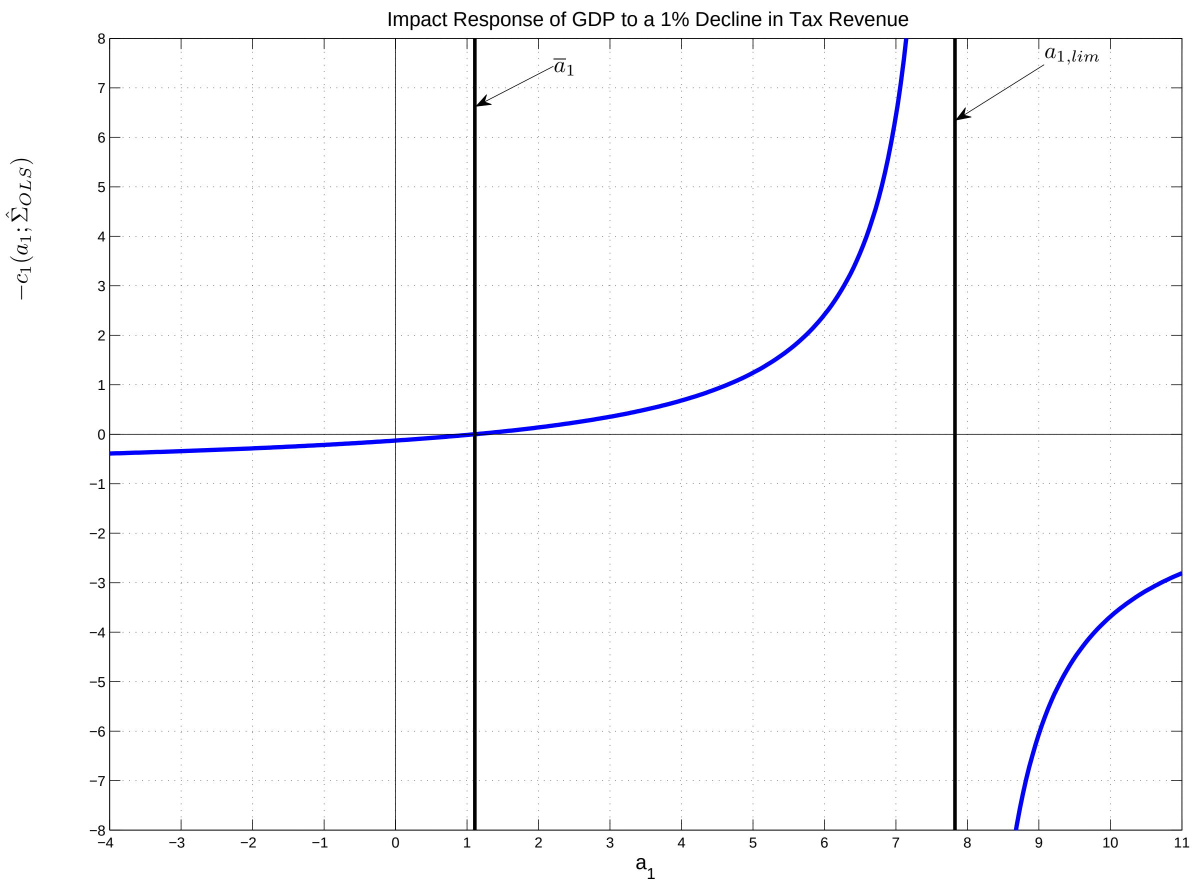

Impact Response of GDP to a 1% Decline in Tax Revenue
$\overline{a}_1$
$a_{1,lim}$
$-c_1(a_1;\hat{\Sigma}_{OLS})$
a_1
8
7
6
5
4
3
2
1
0
−1
−2
−3
−4
−5
−6
−7
−8
−4
−3
−2
−1
0
1
2
3
4
5
6
7
8
9
10
11

Figure 1:

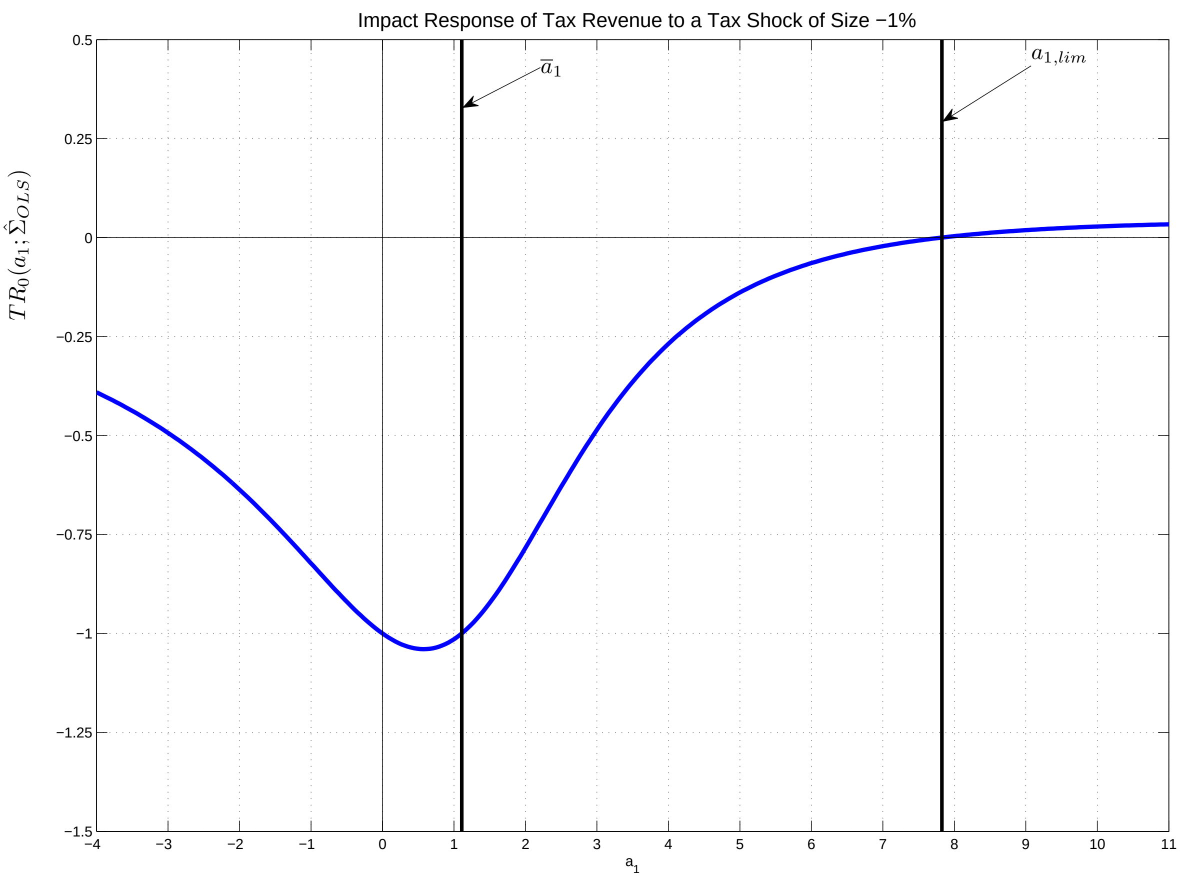

Impact Response of Tax Revenue to a Tax Shock of Size −1%
$\overline{a}_1$
$a_{1,lim}$
$TR_0(a_1;\hat{\Sigma}_{OLS})$
0.5
0.25
0
−0.25
−0.5
−0.75
−1
−1.25
−1.5
−4
−3
−2
−1
0
1
2
3
4
5
6
7
8
9
10
11
a_1

Figure 2:

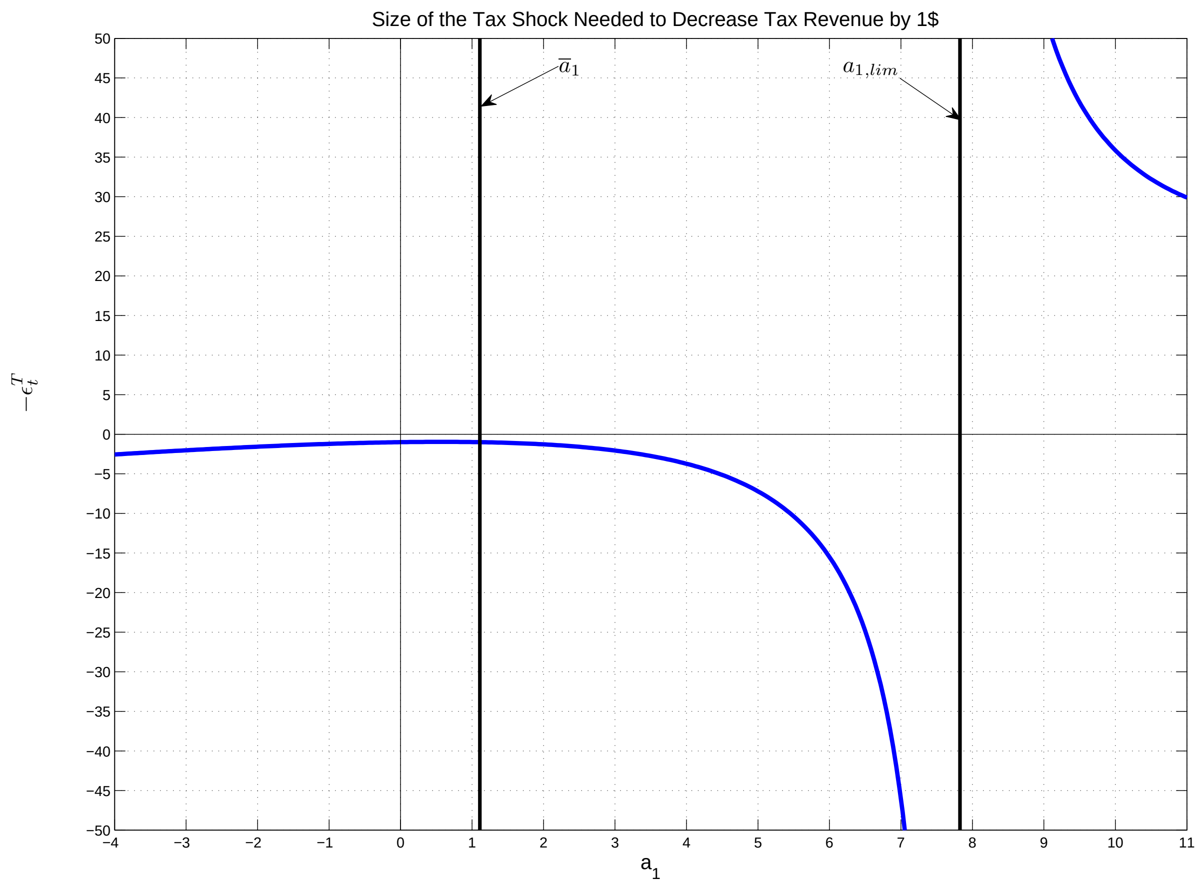
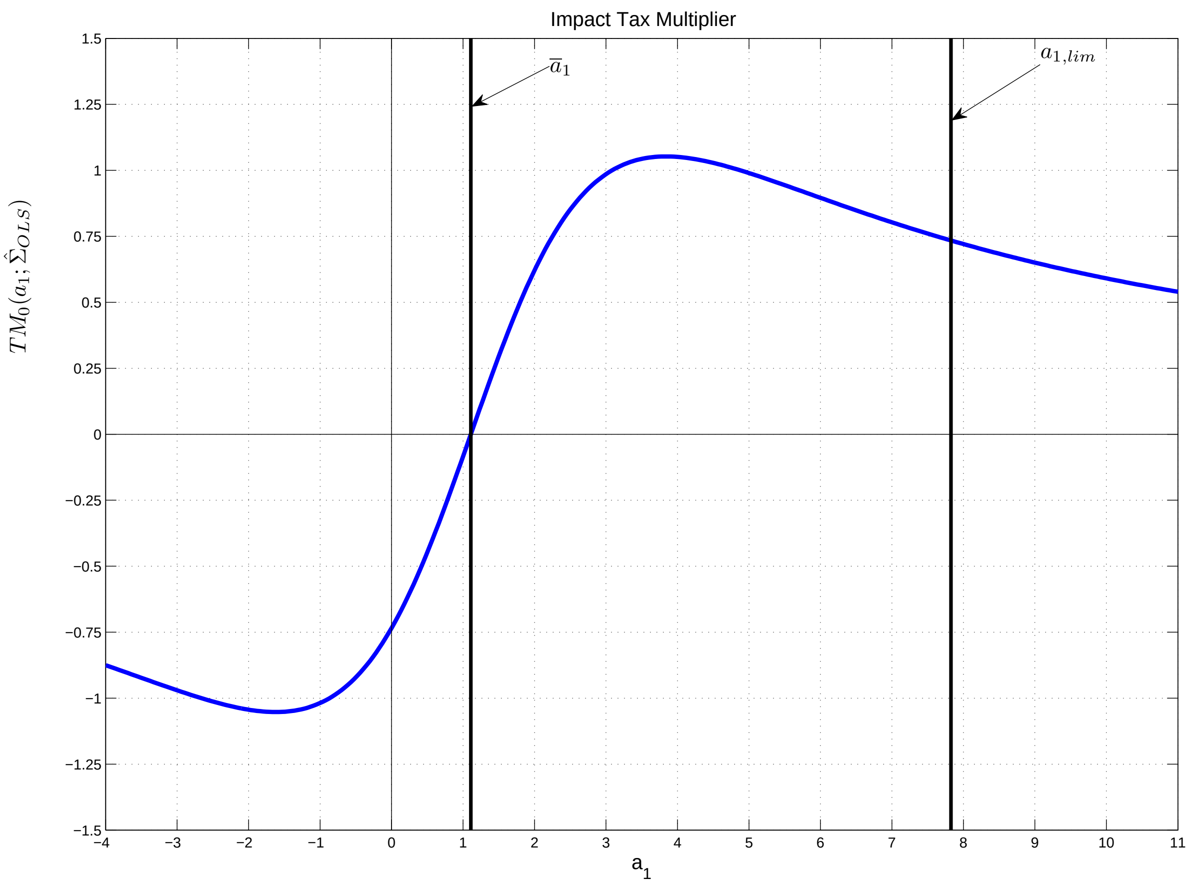

Size of the Tax Shock Needed to Decrease Tax Revenue by 1$
$\overline{a}_1$
$a_{1,lim}$
$-\epsilon_t^T$
a_1
50
45
40
35
30
25
20
15
10
5
0
−5
−10
−15
−20
−25
−30
−35
−40
−45
−50
−4
−3
−2
−1
0
1
2
3
4
5
6
7
8
9
10
11

Figure 3:



Impact Tax Multiplier
$\overline{a}_1$
$a_{1,lim}$
$TM_0(a_1;\hat{\Sigma}_{OLS})$
1.5
1.25
1
0.75
0.5
0.25
0
−0.25
−0.5
−0.75
−1
−1.25
−1.5
−4
−3
−2
−1
0
1
2
3
4
5
6
7
8
9
10
11
a_1

Figure 4:

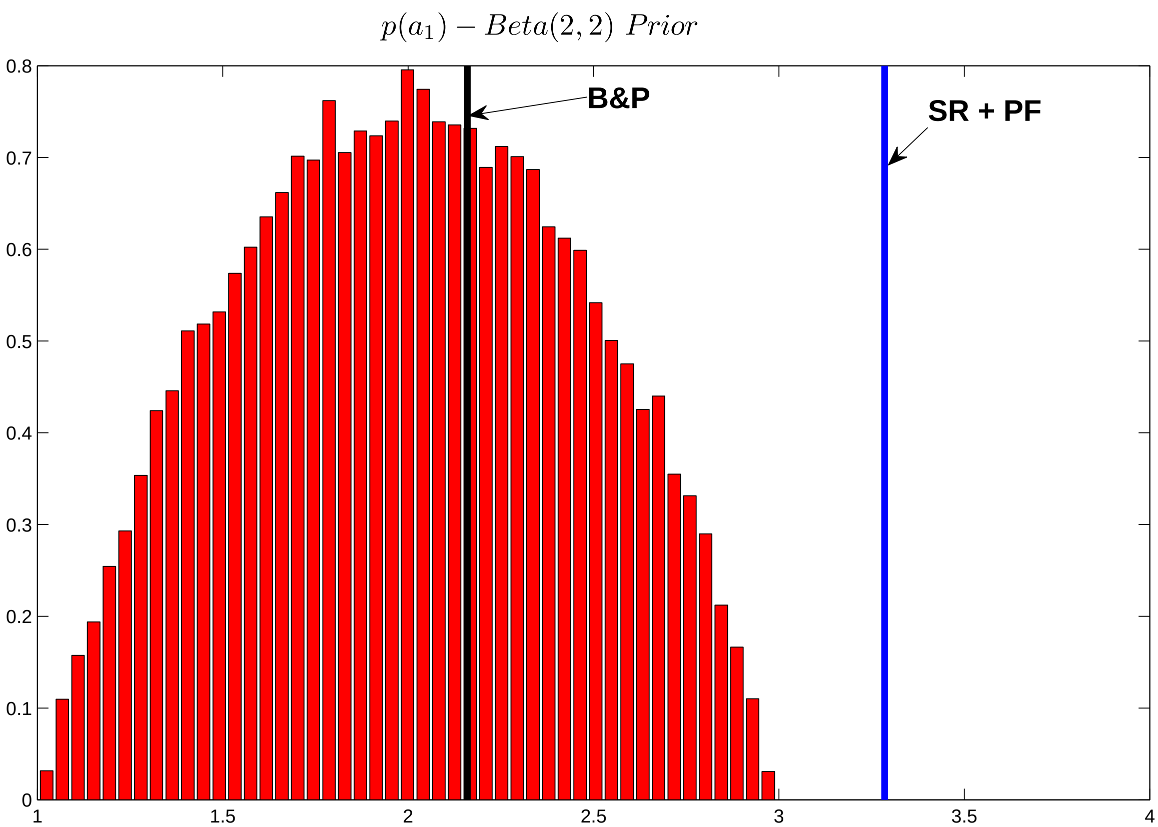
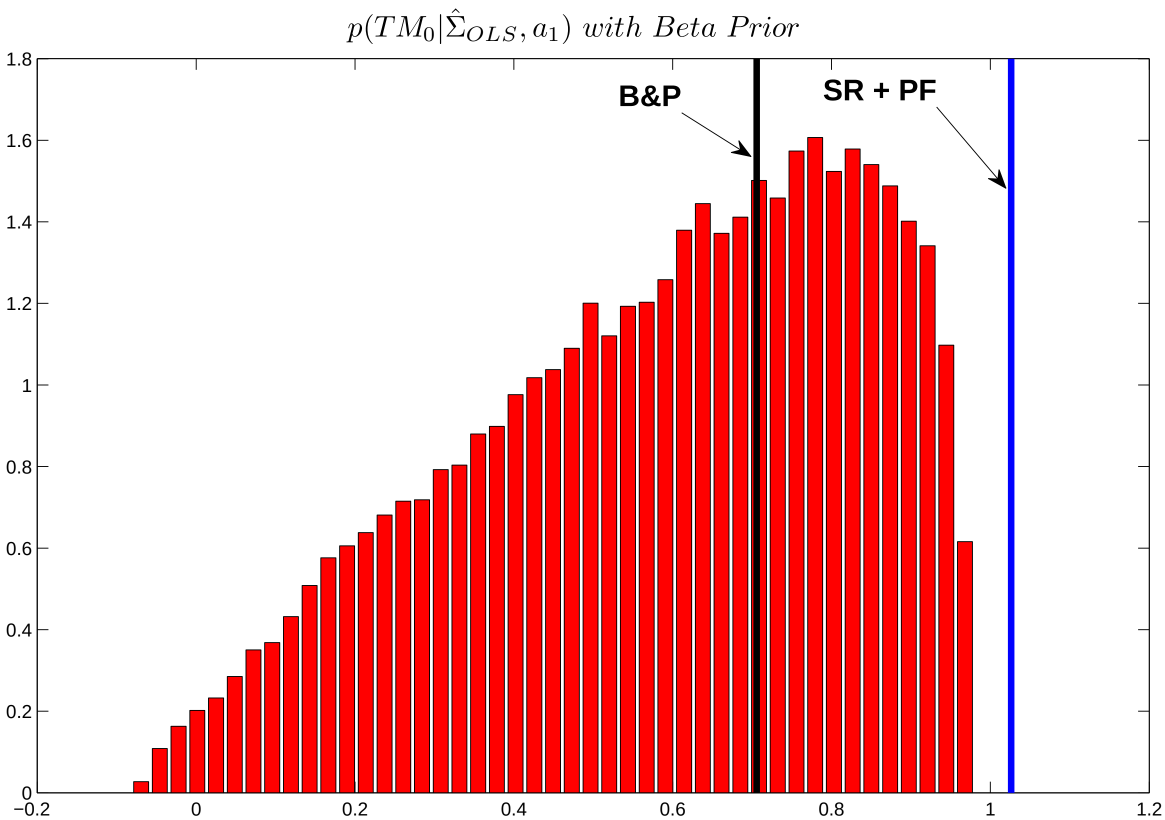


Figure 5:



$p(TM_0|\hat{\Sigma}_{OLS}, a_1)$ with Beta Prior
B&P
SR + PF
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
0
0.2
0.4
0.6
0.8
1
1.2

Figure 6:

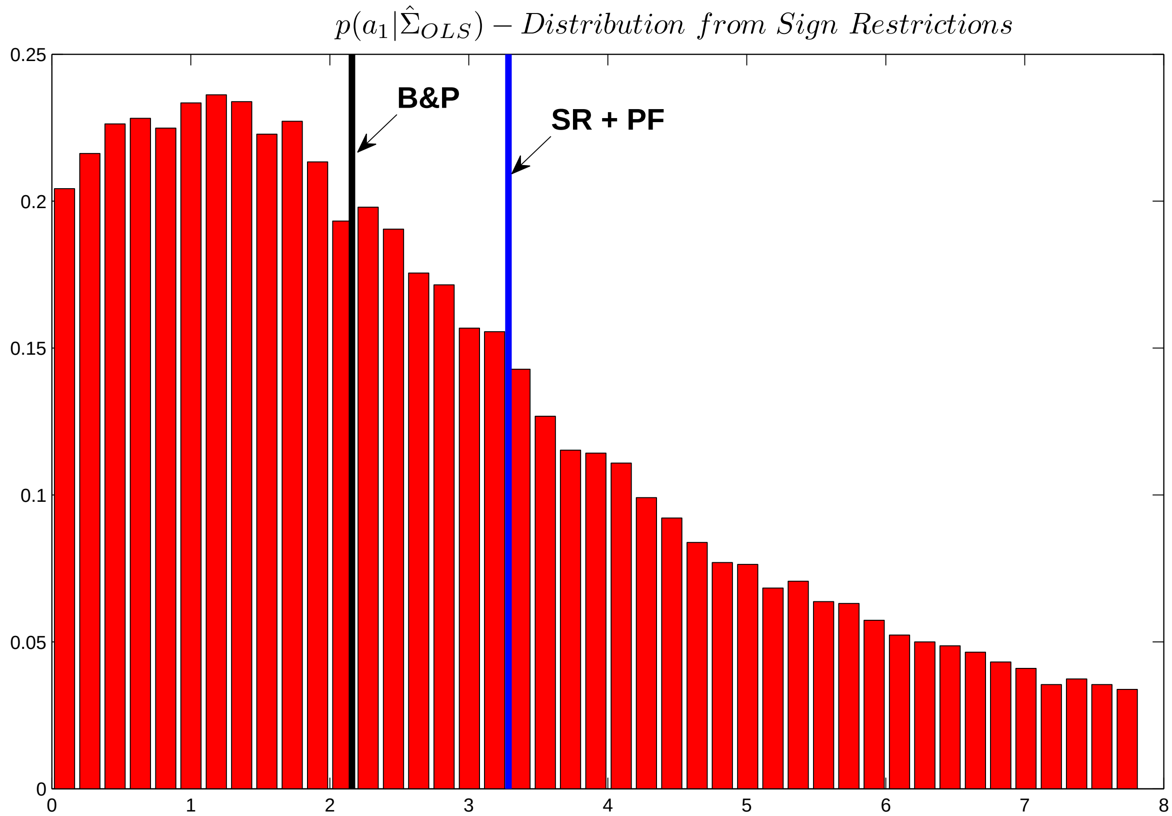

$p(a_1|\hat{\Sigma}_{OLS}) - Distribution\ from\ Sign\ Restrictions$
B&P
SR + PF
0.25
0.2
0.15
0.1
0.05
0
0
1
2
3
4
5
6
7
8

Figure 7:

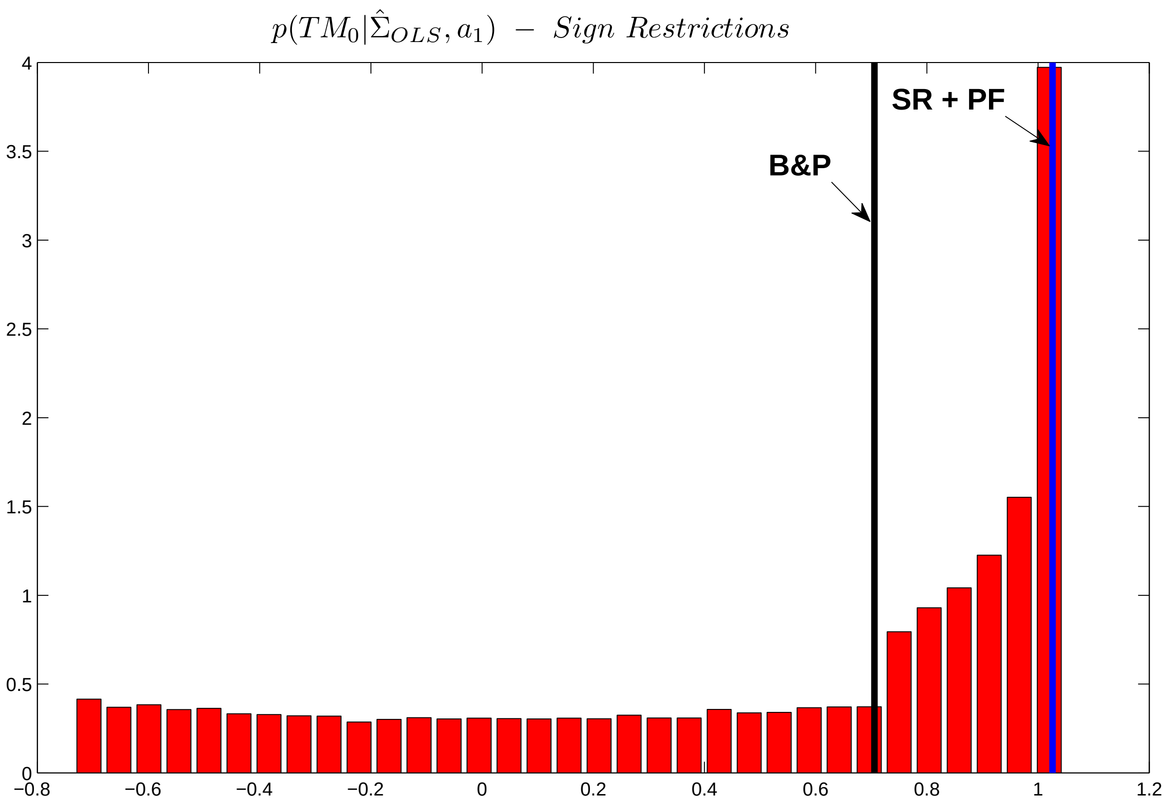

$p(TM_0|\hat{\Sigma}_{OLS}, a_1)$ − Sign Restrictions
B&P
SR + PF
4
3.5
3
2.5
2
1.5
1
0.5
0
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
1.2

Figure 8:

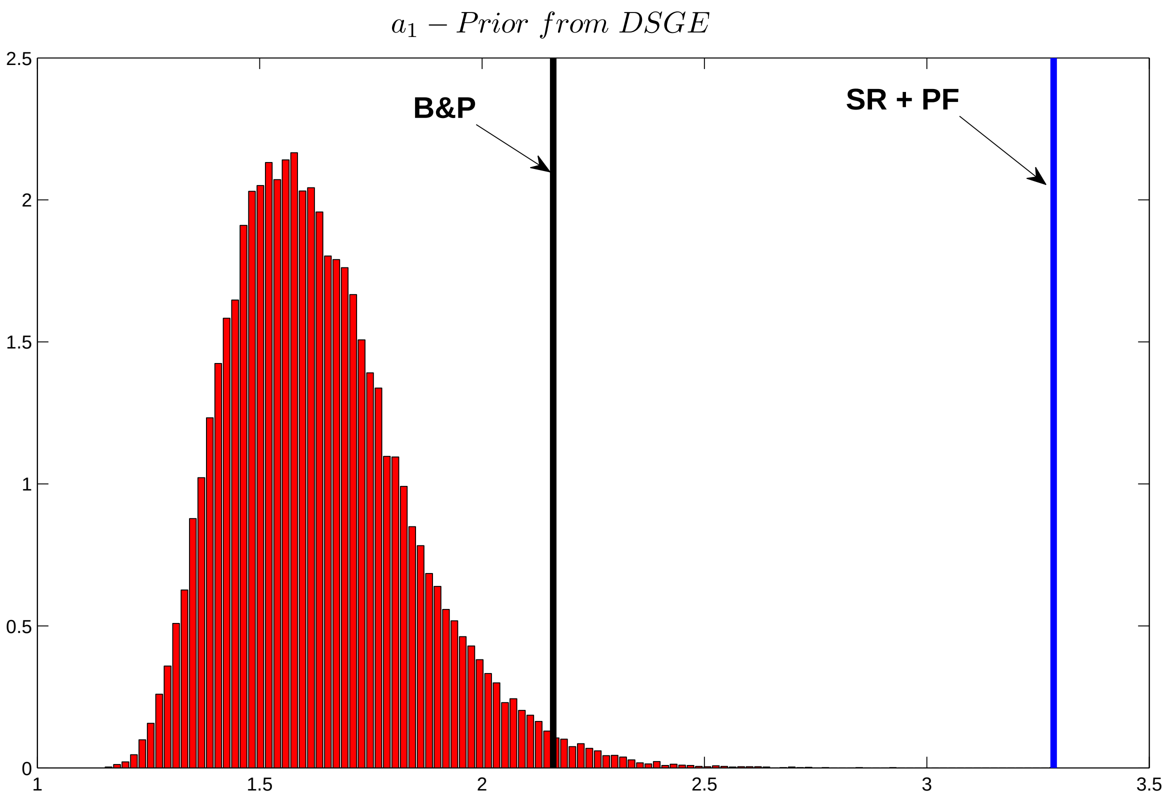


Figure 9:

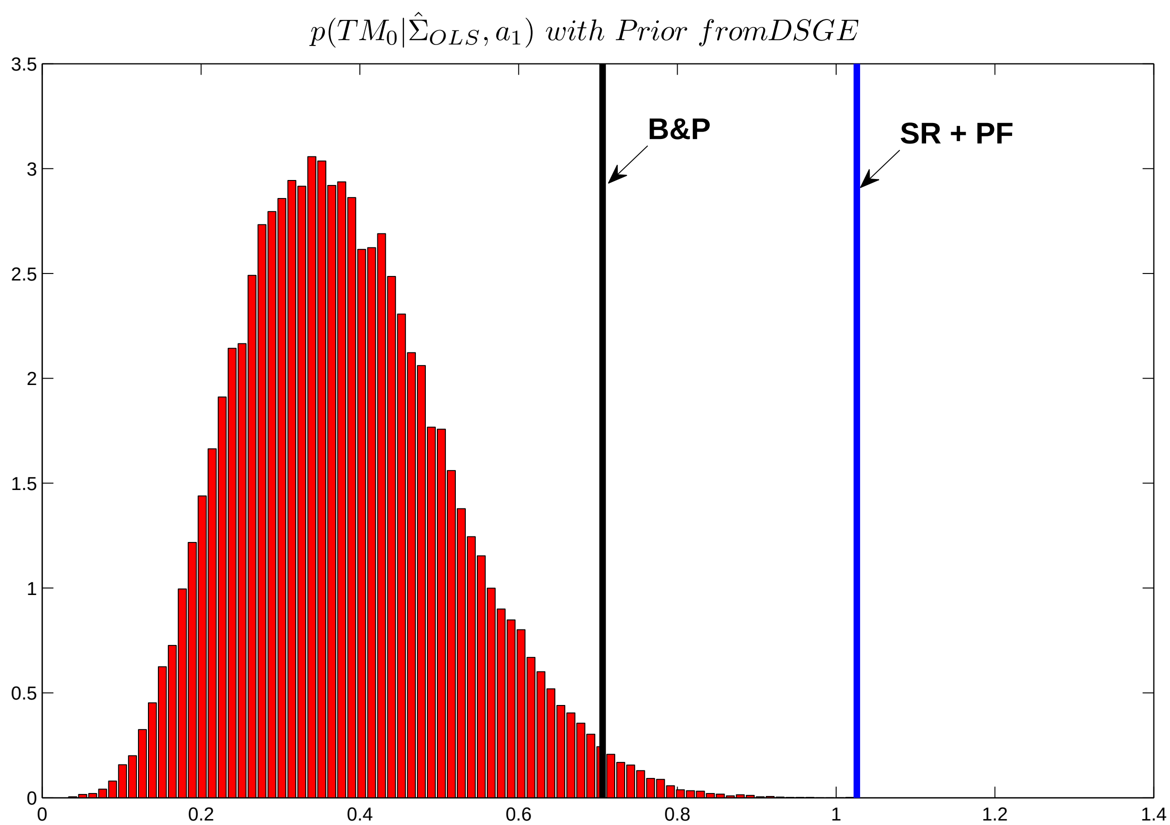

$p(TM_0|\hat{\Sigma}_{OLS}, a_1)$ with Prior fromDSGE
B&P
SR + PF
0
0.5
1
1.5
2
2.5
3
3.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4

Figure 10: