Tax Smoothing, Learning and Debt Volatility*

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Abstract

In this paper I investigate the optimal fiscal policy when markets are complete and private agents are boundedly rational. The main result I find is that the government should use fiscal variables to manipulate agents' expectations. I 'rationalize' the popular view that in periods of pessimism the government should reduce taxes and increase public spending, and vice versa in periods of optimism. Moreover, I can explain some features of tax rate and government debt consistent with the empirical evidence that the complete markets and rational expectations framework cannot match. Finally, I re-examine the validity of some tests for market completeness and debt sustainability in light of my results.

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1 Introduction

"... there is no reason and no occasion for any American to allow his fears to be aroused or his energy and enterprise to be paralyzed by doubt or uncertainty... It is true that the national debt increased sixteen billion dollars... you will be told that the Government spending program of the past five years did not cause the increase in our national income... But that Government spending acted as a trigger, a trigger to set off private activity." . (Franklin Delano Roosevelt, On the Recession (1938))

"Just this week, we learned that retails sales have fallen off a cliff, and so industrial production. All signs point to an economic slump that will be nasty, brutish-and long. How nasty?... the unemployment rate will go above 7 percent, and quite possibly above 8 percent... And how long? It could be very long indeed... there is a lot the federal government can do for the economy... Now it is not the time to worry about the deficit." . (Paul Krugman, (2008))

An important issue in public finance theory is how to collect revenues to pay for government expenditures. When lump-sum transfers are not available, fiscal authorities must resort to taxes which distort people's decisions and move the economy away from the first-best. The optimal taxation literature (e.g. Lucas and Stokey (1983), Chari et al. (1994), Chari and Kehoe (1999)) focuses on identifying the tax profile which minimises the associated distortionary costs. The key insight of this literature is that under complete markets the pay-off of the portfolio of state-contingent bonds works as an insurance device. As a consequence, the tax rate should be smooth and respond very little to shocks.

This conclusion has been derived in a framework in which agents fully understand the problem faced by the government. As they know the problem, they know the solution too, so that their expectations about future tax rates are model-consistent. This requires full information about the model. In many real-world situations the agents' knowledge may not be so deep. Then two questions arise: 1) What happens if the government pursues rational expectations optimal policies but the private sector's expectations are not rational? 2) What is the optimal tax policy if the government recognises that the private sector does not have rational expectations?

The motivation for this work is twofold. The first aspect is normative. As most of the models on optimal taxation assume rational expectations, it is important to check whether they suggest policy recommendations which are robust to alternative expectation formation mechanisms. In this paper I show that the optimal fiscal policy under rational expectations, implemented in a set-up in which agents are boundedly rational, actually generates a sub-optimally high volatility in private consumption and leisure. The second motivating factor is that complete markets models with rational expectations are at odds with the empirical evidence on fiscal variables. In this paper I outline a very simple model that can bridge part of the gap between the data and the theoretical predictions of the complete markets framework.

I consider a closed production economy with no capital and infinitely lived agents. I start assuming that public spending is an exogenous shock, as it is the usual reference point in the public finance literature. Later on I extend the analysis to the case in which the government decides the amount of public consumption. The problem of the household is to maximise her lifetime expected utility subject to her flow budget constraint. The only difference between this framework and the standard optimal fiscal policy one is that agents do not have model-consistent expectations. They act like econometricians and to forecast next period's contingent marginal utility of consumption they use a weighted average of past values of it. Given the realisation of the shock, each period they update their belief about the marginal utility of consumption contingent on that specific realisation.

The government is benevolent and chooses distortionary taxes on labour income and state-contingent debt to maximise households' expected utility, subject to the feasibility constraint, households' optimality conditions and the way in which they update their beliefs.

I find that the government should set fiscal variables to manipulate private agents' expectations. To give an intuition, assume that the public expenditure is constant and that the government has zero initial wealth. Under rational expectations, the optimal fiscal rule prescribes a balanced budget: the government sets the tax rate to collect enough revenues to finance expenditure. When agents do not have rational expectations, this fiscal rule is still feasible, but it will imply a much longer time for agents to learn the tax rate than if the government followed an expectation-dependent fiscal plan. When agents are pessimistic, (i.e. they expect the one-step-ahead tax rate to be higher than they would expect it to be if they were fully rational) government optimality conditions require current expenditure to be financed mainly through debt: in this way the low current tax rate induces agents to revise downwards their expectations about the next period's tax rate.[1] In the long-run the tax rate is higher than in a rational expectations framework because the government has to

[1]One implication of this result is that restricting how much a government can become indebted can delay the learning process.

finance the interest paid on a positive amount of debt.[2]

In this sense the agents' initial beliefs have an effect on the long-run mean value of the tax rate and debt: the more pessimistic (optimistic) the agents are, the higher is the government debt (wealth) in the long-run. One implication of this result is that the model can help explain the wide dispersion across countries in the level of government debt and tax rate.

As expectations are not model-consistent, taxes are less smooth than under rational expectations. The reason is that the government has to minimise the welfare costs associated with distortionary taxes on one hand, and with distorted expectations on the other. When expectations are rational only the first distortion is present, and to minimise the associated losses taxes have to be smooth. But when both distortions are present, this is no longer optimal. The case-study of a perfectly anticipated war is a clear example of the tension between the two conflicting goals the government wants to achieve, tax smoothing on one hand and manipulation of beliefs on the other. Under rational expectations it is optimal for the government to accumulate assets before the war and sell them during the war-time. In this way the tax rate is constant in all periods before and after the war. By contrast, in a learning framework pessimistic agents do not trust the promises made by the government of higher-than-expected future consumption. The government sets low tax rates to manipulate agents' expectations, accumulating less assets (than in a RE framework) before the war. As a consequence, the war is financed issuing more debt than in a RE framework. The tax rate after the big shock is much higher than before.[3]

Since tax rates and debt have a unit-root behaviour, bounded rationality affects the power of some widely used tests to check for market completeness and debt sustainability. In line with Marcet and Scott (2008) I find that looking at the behaviour of debt is a much more reliable way to test the bond market structure than looking at the behaviour of tax rates. Similarly, the standard unit-root test in the debt/GDP ratio used to discriminate between responsible and non-responsible governments can be misleading, since it may cause a fiscal policy plan to be declared unsustainable when instead it is sustainable by construction. Augmenting this test to include the primary surplus in the regressors is a sharper way to distinguish the optimal and sustainable fiscal policy from an unsustainable policy.

Finally, I extend the model to the case in which the government chooses public consumption. I find that, when agents are pessimistic, the fiscal authority increases public spending

[2] The analysis is symmetric for the case of optimistic agents.

[3] Manipulating expectations can explain why a benevolent government should run a deficit during peacetime periods, an implication that the Lucas and Stokey (1983) does not have and for which has been criticized.

above the rational expectations level, financing it mainly through debt. This conclusion is in line with some proposals to deal with the recent distress.

Many authors have studied the impact of learning on monetary policy design, either when the central bank follows some ad hoc policy rules (see *inter alia* Orphanides and Williams (2006), Preston (2005a,b, 2006), Preston and Eusepi (2007b,a)) or when it implements the optimal monetary policy (see *inter alia* Evans and Honkapohja (2003, 2006), Molnar and Santoro (n.d.)). Perhaps surprisingly, fiscal policy has received much less attention. Evans et al. (2007) study the interest rate dynamic learning path in a non-stochastic economy in which the fiscal authority credibly announces a future change in government purchases. Karantounias et al. (2007) and Svec (2008) study the optimal fiscal policy when agents do not trust the transition probabilities of the public expenditure suggested by their approximating model. Up to my knowledge, this is the first paper studying the influence of learning on fiscal policy design.

The paper proceeds as follows. Section 2 studies the consequences of implementing the bond policy function under rational expectations when agents are learning. Section 3 solves for the optimal fiscal policy under learning. In Section 4 I characterise the fiscal plan restricting the government expenditure shock to a specific form. Section 5 gives some policy implications. In section 6 I extend the basic model to the case of endogenous government expenditure. Section 7 deals with the problem of discriminating between a complete markets model with learning and an incomplete markets model with rational expectations. Section 8 focuses on debt sustainability and debt limits. In section 9 I report some stylized facts about fiscal variables and agents' sentiment and use US data in order to test the model. Section 10 concludes.

2 The Model

I consider an infinite horizon economy where the only source of aggregate uncertainty is represented by a government expenditure shock.[4] Time is discrete and indexed by $t = 0, 1, 2....$ In each period $t \geq 0$ there is a realisation of a stochastic event $g_t \in G$. The history of events up and until time t is denoted by $g^t = [g_t, g_{t-1}, g_{t-2}, ..., g_0]$. The conditional probability of g^r given g^t is denoted by $\pi(g^r|g^t)$. For notational convenience, I let $\{x\} = \{x(g^t)\}_{g^t \in G}$ represent the entire state-contingent sequence for any variable x throughout the paper.

[4]In section 6 I extend the analysis to the case in which the fiscal authority chooses the amount of public consumption.

In section 2.1 I briefly review the Lucas and Stokey (1983) model. The economy is populated by a representative household and a government. To finance an exogenous stream of public consumption, the government levies a proportional tax on labour income and has access to a complete set of one-period state-contingent bonds. Both the household and the government have rational expectations. The solution to this model states policy rules for labour tax rate and bond-holdings which maximise households' welfare subject to the restriction that taxes are distortionary. In section 2.2 I assume that, although the government follows the bond-holdings policy rule as in section 2.1 households' expectations are not rational; I show that in this case the optimal fiscal policy under rational expectations translates into sub-optimal volatility of the allocation.

2.1 Rational expectations by both the government and agents

Consider a production economy where the technology is linear in labour. The household is endowed with 1 unit of time that can be used for leisure and labour. Output can be used either for private consumption or public consumption. The resource constraint is

$$c_t + g_t = 1 - l_t \tag{1}$$

where c_t, l_t and g_t denote respectively private consumption, leisure, and public consumption.

The problem of the household is to maximise his lifetime discounted expected utility

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \tag{2}$$

subject to the period-by-period budget constraint

$$b_{t-1}(g_t) + (1-\tau_t)(1-l_t) = c_t + \sum_{g^{t+1}|g^t} b_t(g_{t+1}) p_t^b(g_{t+1}) \tag{3}$$

where β is the discount factor, τ_t is the state-contingent labour tax rate and $b_t(g_{t+1})$ denotes the amount of bonds issued at time t contingent on period $t+1$ government shock at the price $p_t^b(g_{t+1})$. $vb_t \equiv -\sum_{g^{t+1}|g^t} b_t(g_{t+1}) p_t^b(g_{t+1})$ is defined as the value of government debt.

The household's optimality condition are

$$1 - \tau_t = \frac{u_{l,t}}{u_{c,t}} \tag{4}$$

$$p_t^b(g_{t+1}) = \beta \frac{u_{c,t+1}(g_{t+1})}{u_{c,t}} \pi(g^{t+1}|g^t) \quad (5)$$

together with the budget constraint 3.

The government pursues an optimal taxation approach: given an initial amount of inherited debt, b^g_{-1}, she chooses the sequence of tax rates and state-contingent bonds to maximise consumer's welfare. The solution to this dynamic optimal taxation problem is called a Ramsey plan. Lucas and Stokey (1983) show that under complete markets and rational expectations the Ramsey plan has to satisfy the following restriction

$$E_0 \sum_{t=0}^{\infty} \beta^t (u_{c,t} c_t - u_{l,t}(1 - l_t)) = u_{c,0} b_{-1} \quad (6)$$

which can be thought of as the intertemporal consumer budget constraint with both prices and taxes replaced by the households' optimality conditions, (4) and (5). Constraint (6) is the implementability condition. The Ramsey plan satisfies

$$\tau_t = T(g_t, b^g_{-1}) \forall t > 0 \quad (7)$$

$$b^g_t(g_{t+1} = \bar{g}) = D(\bar{g}, b^g_{-1}) \forall t > 0 \quad (8)$$

$$vb^g_t = V(g_t, b^g_{-1}) \forall t > 0 \quad (9)$$

The allocation is a time invariant function of the only state variable in this model, g_t. The initial holding of government bonds matters for the allocation because it determines the value of the Lagrange multiplier attached to the implementability condition. The state-contingent bond holding is a time invariant function and does not depend on the current state of the economy, and the market value of debt is influenced by the current shock only through variations in the state-contingent interest rates.

2.2 The government behaves as in Lucas and Stokey (1983) but agents are boundedly rational

I assume now that agents are learning fiscal policy: they have perfect knowledge about their own decision problem, in the sense that they correctly understand their own objective function and constraints, but they do not know the problem that the other agents in the economy, including the government, have to solve.

The representative agent's problem is to maximise the lifetime utility

$$\tilde{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

subject to the flow budget constraint equation (3). $\tilde{E}_0$ denotes the agent's subjective expectations.

Given the tax rate, equation (4) gives the combination of consumption and leisure. It is still open how much the agent consumes and saves. Given the price of the state-contingent bond, the inter-temporal optimality condition

$$p_t^b(g_{t+1}) = \beta \frac{\tilde{u}_{c,t+1}(g_{t+1})}{u_{c,t}} \pi(g^{t+1}|g^t) \tag{10}$$

.

dictates that the optimal consumption choice depends on the forecast of next-period marginal utility of consumption. Equation (10) looks very similar to equation (5), with the only difference that now $\tilde{u}_{c,t+1}(g_{t+1})$ is the non-rational expectation, conditional on the information up to time t, about next-period state-contingent marginal utility of consumption.

For simplicity, and to be consistent with the analysis carried out in the rest of the paper, I restrict the government expenditure shock to follow a 3-state Markov chain with the following transition probabilities

$$P = \begin{pmatrix} \pi_{L,L} & \pi_{L,M} & \pi_{L,H} \\ \pi_{M,L} & \pi_{M,M} & \pi_{M,H} \\ \pi_{H,L} & \pi_{H,M} & \pi_{H,H} \end{pmatrix}$$

$\pi_{i,j}$ is the probability of moving from state i to state j in one period, for $i = L, M, H$ and $j = L, M, H$.

Given the law of motion for the shock, each period agents have to forecast three values, one for each realisation of the shock. Let $\gamma_t^i \equiv \tilde{u}_{c,t+1}(g_{t+1} = g^i)$ for $i = L, M, H$. Beliefs evolve over time according to the following scheme:

$$\gamma_t^i = \begin{cases} \gamma_{t-1}^i + \alpha_t(u_{c,t}(g_t = g_i) - \gamma_{t-1}^i), & \text{if } g_t = g_i \\ \gamma_{t-1}^i, & \text{if } g_t = g_j \end{cases} \tag{11}$$

with $i = L, M, H$. α_t represents the weight of the forecasting errors when updating the estimates. In this section we consider two standard specifications for the gain α_t, namely $\alpha_t = \frac{1}{t}$ and $\alpha_t = \alpha$.

I assume that the government implements the policy sequences $\{b_{t+j}(g_{t+j+1})\}_{j=0}^{\infty}$ coming out of (8). The time-line of the events is the following. At the beginning of period t, agents observe the realisation of the shock g_t and the tax rate set by the government at t: using all the information up to $t-1$, they form their expectations about next-period state-contingent marginal utility of consumption and decide how much to consume.

The question is whether, given a sufficient amount of data, this equilibrium converges to the rational expectations equilibrium. The criterion adopted to judge convergence under a recursive least-squares algorithm is the expectational stability of rational expectations equilibrium, called E-stability by Evans and Honkapohja (2001). The following proposition holds:

Proposition 2.2.1. *Suppose that the government does not make the fiscal plan contingent on agents' beliefs and follows the bond fiscal rule which is optimal under rational expectations. Assume that agents' utility is separable into consumption and leisure, logarithmic in consumption and linear in leisure. Assume moreover that the initial bond holding is zero. Then the rational expectations equilibrium is learnable.*

Proof. We relegate the proof to the appendix. □

Even when the rational expectation equilibrium is learnable, along the transition path, the fluctuations in the allocation can be quite large with respect to those under rational expectations. Conditioning on the government implementing the rational expectations bond policy rule, we compute the standard deviation of consumption and leisure when agents learn and when they have rational expectations. To do this we simulate the model.

We assume the utility function

$$u(c_t, l_t) = \frac{c_t^{1-\sigma}}{1-\sigma} + log(l_t) \tag{12}$$

and set $\beta = 0.95$, $g_L = 0$, $g_M = 0.1$, $g_H = 0.2$, $\pi_{i,i} = 0.94$, $\pi_{i,j} = 0.03$ $\forall i, j = L, M, H$.

For this parametrisation, the ratio of the standard deviation of consumption when agents are learning to the one when agents have rational expectations is equal to 4.5. This value is averaged across 50000 simulations. The intuition for this result hinges on the implication of tax smoothing in terms of bond portfolio management. To smooth taxes over time and across states the government holds sizable state-contingent bond positions. This policy generates large wealth variation that in turns magnifies the effect of distorted beliefs. The more correlated the shock, the larger the state-contingent bond holdings, and the more distorted beliefs reflect into consumption volatility.

The main implication is that "teaching a lesson", in the sense of imposing the optimal rational expectations policy, is not a good recipe for welfare maximisation. Implementing a fiscal policy plan without taking into account the way in which not fully rational agents form their expectations induces a stream of consumption and leisure that is much more volatile than if agents formed model-consistent expectations. Making the policy plan contingent on agents' beliefs is superior, in terms of welfare, than obstinately performing the optimal fiscal policy under the rational expectations paradigm.

The analysis also raises some doubts about recent developments in the debt management literature, according to which a government should hold extreme debt positions. The first reason to hold such positions relates to the possibility of recovering the complete markets outcome; almost every equilibrium allocation under complete markets can be replicated through an appropriate position in non-contingent bonds at different maturities.[5] Since the welfare is higher under complete markets than under incomplete markets, the optimal debt management is the one that allows the complete markets outcome to be reached. Due to the low variability of next-period bond prices across realisations of shock today, Angeletos (2002) and Buera and Nicolini (2004) show that the government can implement the complete markets allocation holding very extreme positions in bonds with different maturities.

The second reason for debt volatility is to make the full commitment solution time consistent. Persson et al. (1987) consider a model with nominal frictions where the government in charge has an incentive to engage in a surprise inflation to erode the nominal inherited debt. The unique maturity structure of debt implementing the full commitment solution requires the government in t to leave to its successor a positive amount of nominal bond holdings.

[5]Under incomplete markets with only one period bonds, it is not possible to achieve the complete market solution because the implementability condition is replaced with an infinite sequence of period-by-period budget constraints. Aiyagari et al. (2002) for example show that in case of i.i.d. shock, implementing the complete market solution, which generates i.i.d. primary surplus, would imply an explosive path for debt.

3 Optimal Fiscal Policy When Agents Are Learning

The previous section shows that the optimal fiscal policy under rational expectations can lead to very bad outcomes under learning. This suggests that learning should be incorporated in the design of optimal fiscal policy: in this section we study the problem of a government which internalises the fact that agents do not have rational expectations.

The households' optimality condition, which we repeat for convenience, are

$$\frac{u_{l,t}}{u_{c,t}} = 1 - \tau_t \tag{13}$$

$$p_t^b(g_{t+1}) = \beta \frac{\tilde{u}_{c,t+1}(g_{t+1})}{u_{c,t}} \pi(g^{t+1}|g^t) \tag{14}$$

The implementation of equation (14) requires agents to forecast their own state-contingent consumption one-period-ahead. This approach of modeling boundedly rational behaviour may seem strange at first glance, but it is commonly used in the learning literature (see Evans et al. (2003), Carceles-Poveda and Giannitsarou (2007), Milani (2007) among many others). It is a very useful short-cut to model households' lack of knowledge about market determined variables, which are outside of agents' control although they are relevant to their decision problem. In the current setup, non-rational expectations about future consumption can be interpreted as non rational expectations about the tax policy rule followed by the government. In fact, considering the next-period equivalent of equation (13), agents understand that consumption at $t+1$ depends on the tax rate the government will set at $t+1$; as far as expectations about tax rate are not-model consistent, expectations about consumption are neither.

We simplify even further the analysis in section 2.2 assuming that the government expenditure shock can take only two realisations, g_H and g_L, with $g_H ¿ g_L$. Let $\gamma_t^i \equiv \tilde{u}_{c,t+1}(g_{t+1} = g^i)$ for $i = H, L$. As before, agents update their beliefs according to the following scheme

$$\gamma_t^i = \begin{cases} \gamma_{t-1}^i + \alpha_t(u_{c,t}(g_t = g_i) - \gamma_{t-1}^i), & \text{if } g_t = g_i \\ \gamma_{t-1}^i, & \text{if } g_t = g_j \end{cases} \tag{15}$$

with $i = H, L$ and where α_t follows an exogenous law of motion.[6]

Definition 1. *A competitive equilibrium with boundedly rational agents is an allocation* $\{c_t, l_t, g_t\}_{t=0}^{\infty}$*, state-contingent beliefs about one-step-ahead marginal utility of consumption* $\{\gamma_t^i\}_{t=0}^{\infty}$ *for* $i = H, L$*, a price system* $\{p_t^b\}_{t=0}^{\infty}$ *and a government policy* $\{g_t, \tau_t, b_t\}_{t=0}^{\infty}$ *such that*

[6] In Appendix A.10 we discuss a measure of the 'quality' of this learning scheme.

(a) given the price system, the beliefs and the government policy the households' optimality conditions are satisfied; (b) given the allocation and the price system the government policy satisfies the sequence of government budget constraint (3)*; and (c) the goods and the bond markets clear.*

Let

$$x_t = [\gamma_t^H I(g_{t+1} = g_H) + \gamma_t^L I(g_{t+1} = g_L)] \tag{16}$$

where I is the indicator function and define

$$A_t \equiv \prod_{k=0}^{t} \frac{x_{k-1}}{u_{c,k}} \tag{17}$$

Taking logs to both sides we get

$$logA_t = \sum_{k=0}^{t} (log(x_{k-1}) - log(u_{c,k})) \tag{18}$$

The log of A_t is the sum of the log-differences between expected and actual marginal utility of consumption from period 0 to period t. This variable has a very natural interpretation as the sum of all past forecast errors agents have made up to period t in predicting next-period log consumption. Under rational expectations, this variable is constant and equal to 1, while under learning it is not, unless the initial beliefs coincide with the rational expectations ones.

Using households' optimality conditions to substitute out prices and taxes from the government budget constraint, Lucas and Stokey (1983) show that under complete markets and rational expectations the competitive equilibrium imposes one single intertemporal constraint on allocations. Using a similar argument, we show that under complete markets and bounded rationality the following result holds.

Proposition 3.0.2. *Assume that for any competitive equilibrium $\beta^t A_t u_{c,t} \to 0$ a.s.*[7] *Given b_{-1}, γ_{-1}^H, γ_{-1}^L, a feasible allocation $\{c_t, l_t, g_t\}_{t=0}^{\infty}$ is a competitive equilibrium if and only if the following constraint is satisfied*

$$E_0 \sum_{t=0}^{\infty} \beta^t A_t (u_{c,t} c_t - u_{l,t}(1 - l_t)) = A_0 u_{c,0} b_{-1} \tag{19}$$

with initial condition $A_{-1} = 1$

[7]Using the results of Proposition 4.1.2 we show that this is actually the case.

Proof. We relegate the proof to the appendix. □

Equation (19) is the bounded rationality version of the intertemporal constraint on the allocation derived by Lucas and Stokey (1983) in a rational expectations framework. The difference between equations (19) and (6) arises through the effect that out-of-equilibria expectations exert on state-contingent prices. As expectations are not model-consistent, the primary surplus at time t, expressed in terms of marginal utility of consumption, is weighted by the product of ratios of expected to actual marginal utility from period 0 till period t.

3.1 The government problem

Using the *primal approach* to taxation we recast the problem of choosing taxes and state-contingent bonds as a problem of choosing allocations maximising households' welfare over competitive equilibria. At this point a clarification is needed. When the households and the benevolent government share the same information, they maximise the same objective function. But when the way in which they form their expectations differ, as in this setup, their objective functions differ as well. Therefore it is no longer obvious which objective function the government should maximise. In what follows we assume that she maximises the representative consumer's welfare *as if* he were rational. Two reasons justify this assumption. First, as agents form model-consistent expectations in the long-run, in the long-run agents are going to be rational. Second, the government understands how agents behave and form their beliefs, and it understands that these beliefs are distorted. Consequently, it uses this information to give the allocation which is best for them from an objective point of view. This is consistent with a paternalistic vision of the government.[8]

Definition 2. *The government problem under learning is*

$$\max_{\{c_t,l_t,\gamma_t^H,\gamma_t^L,A_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

subject to

$$E_0 \sum_{t=0}^{\infty} \beta^t A_t (u_{c,t} c_t - u_{l,t}(1 - l_t)) = A_0 u_{c,0} b_{-1} \tag{20}$$

$$A_t = A_{t-1} \frac{\gamma_{t-1}^H I(g_t = g^H) + \gamma_{t-1}^L I(g_t = g^L)}{u_{c,t}} \tag{21}$$

[8] The same assumption is made in Karantounias et al. (2007).

$$\gamma_t^i = \begin{cases} \gamma_{t-1}^i + \alpha_t(u_{c,t}(g_t = g_i) - \gamma_{t-1}^i), & \text{if } g_t = g_i \\ \gamma_{t-1}^i, & \text{if } g_t = g_j \end{cases} \tag{22}$$

$$c_t + g_t = 1 - l_t \tag{23}$$

Equation (20) constraints the allocation to be chosen among competitive equilibria. Equation (21) is the recursive formulation for A_t, obtained directly from equation (17). Equation (22) gives the law of motion of beliefs. Equation (23) is the resource constraint. Since A_t and γ_t^i for $i = L, H$ have a recursive structure, the problem becomes recursive adding A_{t-1} and γ_{t-1}^i for $i = L, H$ as state variables.

Leaving the details about the derivation in appendix A.6, first order necessary conditions[9] with respect to consumption and leisure impose that

$$\begin{aligned} &u_{c,t} + \Delta A_t(u_{cc,t}c_t + u_{c,t}) - \lambda_{1,t}\alpha_t u_{cc,t} I(g_t = g^H) \\ &- \lambda_{2,t}\alpha_t u_{cc,t} I(g_t = g^L) - \Delta \frac{u_{cc,t}}{u_{c,t}} E_t \sum_{j=0}^{\infty} \beta^j A_{t+j}(u_{c,t+j}c_{t+j} - u_{l,t+j}(1 - l_{t+j})) = \lambda_{3,t} \end{aligned} \tag{24}$$

$$u_{l,t} + \Delta A_t(u_{l,t} - u_{ll,t}(1 - l_t)) = \lambda_{3,t} \tag{25}$$

The first term on the left side of equation (24) represents the benefit for the government from increasing consumption by one unit. The second one measures the impact of the implementability constraint on the allocation, weighted by the distortion A_t represented by non rational expectations. The third and fourth terms reflect the fact that the government takes into account how agents update their expectations on the basis of the current consumption. The last term on the left represents the derivative of all future expected discounted primary surpluses with respect to current consumption. This is because from equation (19) each primary surplus (in terms of marginal utility) at $t + j, \forall j \geq 0$ is pre-multiplied by the product of past ratios of expected to actual marginal utility. In choosing optimal consumption today, the government is implicitly choosing the factor at which all future primary surpluses are discounted through its effect on A_t. The term on the right is the shadow value of output. A similar interpretation holds for the optimality condition with respect to leisure, equation (25).

Several comments are necessary. First, the optimal allocation is history-dependent through the presence of A_{t-1}: differently from Lucas and Stokey (1983), the allocation is not any

[9] As standard in the optimal fiscal policy literature, it is not easy to establish that the feasible set of the Ramsey problem is convex. To overcome this problem in our numerical calculations we check that the solution to the first-order necessary conditions of the Lagrangian is unique.

more a time-invariant function of the current realisation of the government shock only, but depends on what happened in the past. Second, in appendix A.7 we show that the optimality conditions in a complete markets and rational expectations framework are a special case of equation (24) and (25). Third, using the recursive formulation of A_t, the intertemporal budget constraint at t

$$b_{t-1}A_t u_{c,t} = E_t \sum_{j=0}^{\infty} \beta^j A_{t+j}(u_{c,t+j}c_{t+j} - u_{l,t+j}(1 - l_{t+j}))$$

and combining (24) and (25) the optimal allocation satisfies the following equations:

$$\begin{aligned} u_{c,t} + \Delta A_t(u_{cc,t}(c_t - b_{t-1}) + u_{c,t}) - \lambda_{1,t}\alpha_t u_{cc,t} I(g_t = g^H) - \lambda_{2,t}\alpha_t u_{cc,t} I(g_t = g^L) = \\ u_{l,t} + \Delta A_t(u_{l,t} - u_{ll,t}(1 - l_t)) \end{aligned} \tag{26}$$

Equation (26) looks very similar to the first-order condition with respect to consumption in the incomplete markets model of Aiyagari et al. (2002). In facts in both frameworks the excess burden of taxation is not constant, although for different reasons. In the absence of a full set of state-contingent bonds, as in Aiyagari et al. (2002), the excess burden of taxation is time-varying because of the incomplete insurance offered by the financial bonds: since the interest payment on last period debt is fixed across realisations of the current government shock, the government in each period has to adjust the stream of all future taxes to ensure solvency.[10] In a complete markets model with learning, what makes the excess burden of taxation time-varying is the cost of issuing state-contingent debt. Although market completeness implies that in each period the government can fully insure against expenditure shocks, the state contingent interest rates change as time goes by because agents' expectations change. When agents stop updating their beliefs because the forecast error is zero, $A_{t+j} = A_{t-1} \forall j \geq 0$, and the excess burden of taxation becomes constant again.

Equation (24) expresses the actual marginal utility of consumption as a function of agents' beliefs about it. Figure 1 shows this relation for a log-log utility function and a given value of A_{t-1}.[11] The left panel displays the actual marginal utility of consumption contingent on the government expenditure shock being low (average with respect to the expected marginal utility of consumption contingent on the government expenditure shock being high), as a function of the previous period belief, γ^L_{t-1}. The right panel display the same for the government expenditure shock being high. Figures 2 and 3 show the tax rate and the state-contingent bond policy functions which guarantee that the convergence between actual and

[10] In a complete market framework with rational expectations the excess burden of taxation is constant because the variable which adjusts to ensure solvency is the pay-off of the portfolio of contingent bonds.

[11] The shape of the mapping is robust to different values of this variable.

expected marginal utility holds. The tax rate is a decreasing function of the previous period expected marginal utility; symmetrically, the state-contingent bond is an increasing function of it.

4 Some examples

In order to characterise the optimal fiscal policy in the framework we are studying, in what follows we consider some examples restricting the government expenditure shock to a specific form.

4.1 Constant government expenditure

Consider the case in which the government expenditure is known to be constant and the initial amount of bond holdings is zero. The Lagrangian associated to the government problem is

$$\begin{aligned} \mathcal{L} = \sum_{t=0}^{\infty} \beta^t [u(c_t, l_t) + \Delta A_t (u_{c,t} c_t - u_{l,t}(1 - l_t)) \\ + \lambda_{1,t}(\gamma_t - (1 - \alpha_t)\gamma_{t-1} - \alpha_t u_{c,t}) + \lambda_{2,t}(1 - l_t - c_t - g)] - \Delta A_0 u_{c,0} b_{-1} \end{aligned} \tag{27}$$

where the notation is the same as before and $x_t = \gamma_t$.

The optimality conditions $\forall t \geq 0$ are:

$$\begin{aligned} & u_{c,t} + \Delta A_t (u_{cc,t} c_t + u_{c,t}) - \lambda_{1,t} \alpha_t u_{cc,t} - \\ & \Delta \frac{u_{cc,t}}{u_{c,t}} \sum_{j=0}^{\infty} \beta^j A_{t+j} (u_{c,t+j} c_{t+j} - u_{l,t+j}(1 - l_{t+j})) = u_{l,t} + \Delta A_t (u_{l,t} - u_{ll,t}(1 - l_t)) \end{aligned} \tag{28}$$

$$\lambda_{1,t} - \beta(1 - \alpha_t)\lambda_{1,t+1} + \beta \Delta b_t A_t = 0$$

Equation (28) gives the mapping T between agents' beliefs about (marginal utility of) consumption and actual (marginal utility of) consumption. In the next proposition we characterise the properties of this mapping.

Proposition 4.1.1. *Assume the utility function*

$$u(c_t, l_t) = log c_t + l_t \tag{29}$$

and that the gain α_t is small enough.

Then, in the set $\gamma_{t-1} > 0$ the mapping T: $R_+ \to R_+$ has the following properties:

- *T is increasing and concave.*
- *T has one fixed point.*
- *The least squares learning converges to it.*

Proof. We relegate the proof to the appendix. □

Proposition 4.1.1 shows that the expected marginal utility converges to actual one, so that in the long-run agents' expectations are model-consistent and the forecast error is zero. The next proposition characterises the value towards which expectations converge.

Proposition 4.1.2. *Given an initial value for the government bond holding b_{-1} the allocation under learning does not converge to the allocation under rational expectations implied by the same initial bond holding. However, for any initial belief held by agents, there exists a b_{-1} such that*

$$lim_{t\to\infty} c_t^L(\gamma_{-1}) = c_t^{RE}(b_{-1}) \tag{30}$$

Proof. We relegate the proof to the appendix. □

For any initial belief about the marginal utility of consumption, there is always an initial level of government wealth such that the allocation under learning converges to the one under rational expectations starting with that initial government wealth. Figure 4 shows this relation assuming that in equation (27) $b_{-1} = 0$. Given the parameters values used, the solution of the Ramsey problem under rational expectations implies that the marginal utility of consumption is constant and equal to 2.5. For all values of initial belief higher (lower) than this reference value, the learning allocation coincides with the solution of a Ramsey problem in an economy populated by rational agents and endowed with a positive (negative) initial government debt.

4.1.1 Policy implications

The example of constant government consumption highlights the impact of expectations on the optimal fiscal plan. Under rational expectations, the only distortion is the one associated to taxes. In order to smooth this distortion over time, taxes are set to balance the government budget every period. In this way agents can enjoy a perfectly constant allocation. By contrast under learning, there are two distortions in the economy, one associated with taxes and the other one associated with agents' expectations. Therefore, although the government could follow a balanced-budget rule, it decides not to do it because in this way it would not minimise the *overall* distortions. To influence out-of-equilibria expectations the government animates initially pessimistic agents setting a low tax rate at the beginning and financing the public consumption with debt. As time goes by, the tax rate has to increase in order to ensure government solvency.[12]. This stabilization policy is resistant to a selection of robustness checks. For example, it holds if 1) we suppose that agents use lagged value of marginal utility of consumption to update their current beliefs, 2) the government has access to consumption taxes instead of labour ones.

Figures 5, 6 and 7 offer a graphical interpretation of the result. The solid lines show the optimal fiscal plan under rational expectations. Whereas the dashed lines show the optimal fiscal plan when agents adopt a constant gain algorithm to update their belief while supposing that in the initial period the expected consumption is lower than the actual consumption prevailing at $t = 0$.[13]

4.2 A single big shock

Consider the case in which expenditure is constant in all periods apart from T, when $g_T > g_t$. Both the government and households know the entire path of the expenditure, so that the shock in T is perfectly anticipated. Under rational expectations the government runs a positive primary surplus from period 0 to $T-1$, using it to buy bonds. At T the government finances the high public consumption level by selling the accumulated assets and possibly by levying a tax rate on labour income. From period $T+1$ onwards the tax rate is just sufficient to cover the expenditure and to service the interest on the bonds issued at T. By contrast, in an economy populated by pessimistic agents, the government can accumulate less assets because it has to stimulate the economy to manipulate expectations. The big shock at T is financed by increasing debt much more than under rational expectations. Figures 8-10 illustrate the optimal plan under rational expectations (the solid lines) and under learning (the dashed lines), assuming $T = 10$, $g_t = 0.1$ and $g_T = 0.2$.

4.2.1 Policy implications

The example of a perfectly anticipated war is useful for two reasons. First, it clarifies how the tax smoothing result is altered by the presence of boundedly rational agents. Under

[12]The analysis is symmetric for the case of initially optimistic agents

[13]Since for optimistic agents the evolution of the system is symmetric, we do not report it.

rational expectations the government spreads over time the cost of financing the war in T thorough distortionary taxes. As a result, the tax rate is perfectly constant in all periods before and after the war: taxes are smooth in the sense that they have a smaller variance than a balanced-budget rule would imply. By contrast, when agents are learning, they do not trust the promises made by the fiscal authority in terms of future consumption. The government uses taxes and debt to correct agents' distorted expectations, in a way that the tax rate is more volatile than under rational expectations.

The example is relevant also because it reconciles the complete markets framework with the empirical evidence that during peacetime periods countries run a primary deficit. The Lucas and Stokey (1983) model is unable to fit this evidence, as the government runs a primary surplus to accumulate assets before the war.

4.3 Cyclical shocks

Suppose that

$$\begin{aligned} g_{t'} &= g_H \quad \forall t' = j \times H \leq \overline{T} \\ g_t &= g_L \quad otherwise \end{aligned}$$

with $j = 1, 2, ..., \frac{\overline{T}}{H}$. H is the length of time over two subsequent bad shocks and $\overline{T}$ is the last period in which a bad shock can occur. The rational expectations policy recipe is the same as before: the tax rate is constant in all periods when $g_t = g_L$ and increases very little when the bad shock hits the economy, due to the assets the government accumulates during the good shock periods. Under learning with pessimistic agents, before the first realisation of the bad shock the tax rate is lower than under RE and increases between any two subsequent bad shocks, generating resources devoted to reducing debt, which increases whenever the bad shock occurs. After the last bad realisation of the shock, the tax rate falls and then gradually increases over time to ensure intertemporal solvency.

4.4 Bad shock of unknown duration

Suppose that the shock can take two realisations, g_L and g_H, with the following transition probabilities matrix

$$P = \begin{pmatrix} 1 & 0 \\ \pi_{H,L} & \pi_{H,H} \end{pmatrix}$$

where $\pi_{i,j}$ is the probability that tomorrow the shock is in state j, being today in state i, with $g_t = g_H$ at $t = 0$. This example corresponds to an absorbing Markov chain, where the low realisation of the shock is the absorbing state and the high one is the transient state. Under rational expectations, the government finances the bad shocks partly through taxes and partly by issuing debt. Numerical results, not reported here, confirm the role of fiscal policy as stabiliser of expectations: the accumulation of public debt is higher and longer under learning than under rational expectations, the difference being due to the opportunity of inducing the agents to revise their expectations downwards.

4.5 Serially correlated shock

Suppose that the shock can take two realisations, g_L and g_H, with the following transition probabilities matrix

$$P = \begin{pmatrix} \pi_{L,L} & \pi_{L,H} \\ \pi_{H,L} & \pi_{H,H} \end{pmatrix}$$

We set $g_L = 0.05$,$g_H = 0.1$ and $\pi_{H,H} = \pi_{L,L} = 0.8$.[14] As in the previous examples, we assume a discount factor equal to 0.95 and a gain parameter equal to 0.02.[15]

Table 2 summarises some statistics for the allocation and the fiscal variables under rational expectations. Table 3 summarises the same statistics under learning after convergence of beliefs for initially pessimistic and optimistic agents. Reported values are average across 1000 simulations. Comparing the two tables we can observe that in the long run initially pessimistic (optimistic) agents consume less (more) than if they had been rational. This result is in line with the examples in sections 4.1 and 4.2. Since at the beginning consumers were pessimistic, the accumulation of debt necessary to induce them to revise upwards their expectations about consumption requires higher taxes in the long run than under rational expectations. Because of this, consumption is lower and leisure is higher (than under rational expectations). The average primary surplus is higher as well. In this sense we can say that beliefs are self-fulfilling in the long run: the lower is the initial expected consumption, the lower the actual consumption after convergence. Exactly the opposite is true with initially optimistic agents.

Table 4 shows the same statistics for the system during the first 30 periods of transition, under rational expectations and initially pessimistic agents. Although all the endogenous

[14]For the case of i.i.d shock case the results are very similar to those with a serially correlated shock, and therefore they are not reported.

[15]The choice of the updating parameter is not easy because it requires a trade-off between filtering noises and tracking structural changes. Milani (2007) estimates a New-Keynesian model and finds that the best fitting specification has a gain coefficient in a range between 0.015 and 0.03. Orphanides and Williams (2004) find that a value for k in the range $0.01 - 0.04$ fits the expectations data from the Survey of Professional Forecasters better than using higher or lower values. Evans et al. (2007) also use the same value for k.

variables are more volatile before convergence than after convergence, the market value of government debt and the labour tax rate are the most volatile. For example, the tax rate volatility before convergence is double that after convergence. This is due to the fact that the government implements an expectation-dependent fiscal plan. When beliefs are distorted, fiscal variables react to correct this distortion. As time passes and agents' expectations become model-consistent, the government stops using fiscal variables to influence distorted beliefs.

5 Policy Implications

The analysis in section 4 has characterized the optimal fiscal plan that a benevolent government should implement when agents are learning. With respect to the rational expectations framework, bounded rationality introduces a new distortion in the economy. The government takes into account the way in which agents form their expectations and realises that these expectations are distorted. The optimal fiscal plan minimises the distortions associated with both taxes and expectations. Stabilising out-of-equilibria expectations requires setting low taxes when agents are initially pessimistic and high ones when they are initially optimistic. This has a cost and a benefit. The cost is represented by the fact that taxes are less smooth, as the examples in sections 4.1 and 4.2 clarify. But the benefit is that under the expectation-dependent fiscal plan agents learn the tax rate policy rule much faster than under the rational expectations optimal fiscal plan. Figure 11 illustrates this point graphically. The solid line shows the next-period marginal utility of consumption forecast error made by the agents when the government follows the rational expectations recipe compared to the case when she implements the optimal policy under learning, represented by the dashed line. Agents are initially pessimistic and the time period is one year. The lower (than RE) tax rate set at the beginning following the optimal fiscal policy plan induces agents to correct their pessimism much faster than if the fiscal policy suggested by Lucas and Stokey (1983), which is the optimal one under rational expectations, were followed.

The way in which the government should use fiscal variables to manipulate agents' distorted expectations in some sense resembles a standard Keynesian-inspired stabilisation policy. However, it is important to stress that the government should not stimulate economic activity indiscriminately. Actually it is very important to implement the right policy at the right moment. In what follows we show that an expansionary fiscal policy, implemented when agents' expectations require a restrictive one, generates a sub-optimal volatility in the system.

Suppose for simplicity that the public consumption shock is constant and that the government wants to animate the economy when agents are optimistic. To this aim, it implements the following tax-rate rule

$$\tau_t = \begin{cases} \tau_t^{pess}, & \forall t \leq T \\ \xi_t \tau_t^{pess} + (1-\xi_t)\tau_t^{bb}. & \forall t > T \end{cases} \tag{31}$$

According to equation (31) the fiscal authority stimulates the economy till period T setting the tax rate at the (low) optimal level when agents are pessimistic, τ_t^{pess}, and that from T onwards sets the tax rate as a weighted average between this value and the one which raises enough revenues to pay-back both the interests on the inherited debt and the current government expenditure shock, τ_t^{bb}. The weight is given by $\xi_t = k^{t-T}$, with $0 < k < 1$. In order to ensure that the transversality condition is not violated, it is necessary to impose the restriction that the weight ξ_t goes to zero quickly. Otherwise the revenues raised through distortionary taxes would not be enough to finance the interests on the debt that the government has accumulated. Therefore, in order to rule-out Ponzi-schemes, the parameter k is set small enough to ensure that the fiscal plan which belongs to the class of the feasible ones.

The dashed lines in figure 14 show the optimal tax rate and bond holdings when agents are optimistic and the government implements the fiscal policy taking into account that they are optimistic. Whereas the solid lines show the same variables when agents are optimistic and the government implements the rule given by equation 31. The dashed lines in figure 15 show the consumption, leisure and forecast error when agents are optimistic and the government implements the optimal fiscal plan conditioning on this, while the solid lines show the same variables under the wrong stimulus.

Two observations are worth noting. First, the allocation is more volatile under the wrong stimulus, and consequently households' welfare is lower. In order to quantify these losses we utilise numerical methods. We assume the utility function:

$$log(c_t) + log(l_t) \tag{32}$$

and set $\beta = 0.95$, $g = 0.1$, $T = 24$ and $k = 0.7$. Given this parametrisation, the welfare losses in terms of consumption-equivalence units of animating the economy when the optimum requires depressing it is equal to 0.2 percent. Increasing T and/or k increases the welfare losses. Using the same parameters as before but with $T = 25$, the welfare losses become equal to 0.25; similarly with $k = 0.78$ the welfare losses is equal to 0.27. The intuition is that the higher is T and/or the higher is k, the more the fiscal policy is expansionary instead of being restrictive, as it should be since agents are optimistic.

The sub-optimally higher volatility generated by an unjustifiable government's desire to stimulate the economy translates into a longer time for agents to learn. While under the optimal fiscal plan the forecast error is zero after 50 years, under the wrong stimulus it fluctuates much more and it is still not zero after 300 years. In conclusion, setting a low tax rate to stimulate the economy when the opposite is required inefficiently induces instability into the system. While a government can accumulate debt even when it is not necessary, a responsible government will only accumulate it when necessary.

6 Endogenous Government Spending

Up to now we have considered public consumption as a completely exogenous shock. This assumption seems quite restrictive, as governments can decide how much to spend. In order to add some realism to the analysis, in this section we consider the same model as in subsection 4.1 but we allow the government to choose the amount of public spending. We find that this extension corroborates the policy implications outlined in section 4.

We assume the utility function

$$u(c_t, l_t, g_t) = log(c_t) + log(l_t) + \alpha log(g_t)$$

with $\alpha < 1$. The Lagrangian associated to the government problem is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t [u(c_t, l_t, g_t) + \Delta A_t(u_{c,t}c_t - u_{l,t}(1 - l_t)) \\ + \lambda_{1,t}(\gamma_t - (1 - \alpha_t)\gamma_{t-1} - \alpha_t u_{c,t}) + \lambda_{2,t}(1 - l_t - c_t - g_t)] - \Delta A_0 u_{c,0} b_{-1}$$

The dashed lines in figures 12 and 13 show the rational expectations equilibrium, whereas the solid lines show the learning optimal allocation when agents are pessimistic. In line with the previous results, at the beginning the government chooses an expansionary fiscal policy, setting higher public spending than under rational expectations and financing it mainly through debt.

7 Testing Complete Versus Incomplete Markets

In section 3 we have shown that the first order condition with respect to consumption in a complete markets model with learning looks very similar to the one in an incomplete markets model with rational expectations because in both cases, although for different reasons, the excess burden of taxation changes over time.

Assessing whether markets are complete or incomplete is not an obvious issue, since there are theoretical justifications in both directions: while transaction costs and limited commitment push in favour of market incompleteness, the possibility of replicating the complete markets equilibrium through a portfolio of bonds with different maturities favours market completeness. The tests proposed in the literature to discriminate between complete and incomplete markets (see inter alia Scott (2007), Marcet and Scott (2008), and Faraglia et al. (2006)) are based on two discriminating features between the two regimes:

1. Under complete markets fiscal variables (tax rate and market value of debt) inherit the serial properties of the underlying shocks hitting the economy, while under incomplete markets they have a unit-root component.

2. Under complete markets the market value of debt and the primary deficit co-move negatively, while under incomplete markets they co-move positively.

Tests based on the first feature are persistence tests, and those based on the second are impact tests. The aim of this section is to show that the tests belonging to the first category are not able to discriminate between an incomplete markets model and a complete markets model when agents have non-rational expectations: in particular we argue that these tests would be prone to accept the wrong hypothesis that markets are incomplete if, in fact, agents learn and markets are complete. The reason is simply that learning creates persistence in the system.[16]

We replicate persistence tests proposed in Scott (1999, 2007) and Faraglia et al. (2007) to check market completeness. The first test is based on the presence of unit root in the labour tax rate. Assume that the government expenditure shock is stationary; under complete markets the labour tax rate is stationary, while under incomplete markets it contains a unit root. We simulate the model described in Section 3 and we apply the Augmented Dickey

[16] One way to compare persistence under learning and under rational expectations is to look at autoregressions of tax rates in the two frameworks when the shock is *i.i.d.*. Table 5 shows that while under rational expectations the coefficient on the lagged tax rate is close to zero and not statistically significant, under learning it is high and statistically significant.

Fuller test to the tax rate using the first 50 periods of data. Out of 1000 simulations, the probability of accepting the unit-root test, and therefore of concluding (erroneously) that markets are incomplete, is equal to 0.999.

The second test is to estimate whether the excess burden of taxation has a unit root: under rational expectations and complete markets the excess burden of taxation is constant, while under rational expectations and incomplete markets it has a unit root, as shown in Aiyagari et al. (2002). Using the same sample period as before, the probability of accepting the unit-root test, and therefore of concluding (erroneously) that markets are incomplete, is equal to 0.922.

The third test in based on the fact that under complete markets the market value of debt and the primary deficit have the same persistence, while under incomplete markets the first is more persistent than the second. This result does not hold in a boundedly rationality framework. The top panel in Fig. 16 displays the persistence of the debt/GDP ratio, the primary surplus/GDP ratio and the government expenditure shock when agents have rational expectations and markets are complete. The bottom panel displays the same variables when agents are boundedly rational and markets are complete.[17]

Results in figure 16 would induce one to accept, once again erroneously, the hypothesis of market incompleteness.

However a model with complete markets and learning is *not* observationally equivalent to a model with incomplete markets and rational expectations. Actually impact tests, already used in the literature, are able to capture the differences between the two frameworks. Consider for example the co-movement between primary deficit and government debt. Independently of the way in which agents form their expectations, under complete markets this co-movement is negative, while under incomplete markets it is positive, and so is in data. Therefore, it seems important to consider a model with both incomplete markets and learning. We leave this for future research.

To conclude, in line with Marcet and Scott (2008) we find that looking at the behaviour of debt is a much more reliable way to test the bond market structure than looking at the behaviour of tax rate.

[17] To measure the persistence of a variable, say y, we use the k-variance ratio, defined as

$$P_y^k = \frac{Var(y_t - y_{t-k})}{kVar(y_t - y_{t-1})}$$

8 Debt Sustainability and Debt Limits

The literature has recently emphasised the opportunity of imposing limits on the amount of debt a government can accumulate.[18] In a context of non-rational agents the long run market value of debt depends on the initial beliefs held by agents: the higher the initial pessimism in the economy, the higher is the long-run level of debt. Since this debt accumulation is "good", in the sense that it allows for convergence between actual and expected marginal utility, there is not necessarily a correspondence between keeping the debt/GDP ratio low and optimal fiscal policy considerations. Moreover, debt limits may fail to discriminate between "good" and "bad" governments. Consider two countries, hit by the same realisation of the government expenditure shock which differ only as to the vector of initial beliefs. Figure 17 shows the probability that the debt limit (set equal to 60 per cent of steady state GDP) is binding for the two countries conditioning on the fact that each of the two governments implements the optimal fiscal plan taking as given the initial degree of pessimism. Since the first country is populated by less pessimistic agents than the second, the long-run debt level is lower in the first than in the second. But this does not mean that the government in the first country is more responsible than the one in the second just because it accumulated less debt. The only reason for the difference in the long-run debt level is that in the first country initial beliefs were less distorted than in the second, and the government had to intervene less to correct them.

The main advantage of debt constraints is that they are helpful in ensuring sustainability of fiscal policy.[19] Assessments of debt sustainability performed by international institutions are usually based on medium-term simulations (generally 5-10 years) of the debt/GDP ratio. A declining trend in debt/GDP ratio is interpreted as a signal that the government follows a sustainable fiscal policy, whereas an increasing one raises doubts about intertemporal solvency. In a model with boundedly rational agents assessing sustainability is particularly cumbersome, exactly because at the beginning government debt displays a trend.

Suppose that an agency wants to test for the presence of unit root in the debt/GDP ratio, in which case the fiscal policy plan is declared unsustainable. We show that actually this test can perform very poorly if the government follows the optimal fiscal policy plan when agents are learning.

[18] In Chari and Kehoe (2004), debt constraints are beneficial if the monetary authority cannot commit to solve the time inconsistency problem of deflating the nominal debt issued by the fiscal authorities of the member states.

[19] A set of tests has been proposed by the literature to check sustainability, among others by Hamilton and Flavin (1986), Trehan and Walsh (1991) and Bohn (1998).

Suppose that agents in the economy are pessimistic and that to cover the government expenditure shock the government at the beginning uses debt, which is financed through government revenues that are increasing over time. The agency is asked to evaluate the government solvency and to do that it applies an Augmented Dickey Fuller test on the market value of debt using the first 50 periods of observations.

$$\frac{debt_t}{GDP_t} = \alpha + \beta_T^{OLS}\frac{debt_{t-1}}{GDP_{t-1}} + \gamma_T^{OLS}(\frac{debt_{t-1}}{GDP_{t-1}} - \frac{debt_{t-2}}{GDP_{t-2}}) + \epsilon_t \tag{33}$$

Over 1000 simulations, the probability that the agency would declare the fiscal plan to be unsustainable when instead it is sustainable by construction is equal to 0.697.[20] The reason for this is that debt is used to manipulate agents' expectations: since these are persistent, they impart persistence to debt as well. In other words, bounded rationality increases the lack of power of unit root tests. Suppose now that the agency applies the Augmented Dickey Fuller test to the following equation

$$\begin{aligned}(\frac{debt}{GDP})_t = \alpha^{OLS} + \beta_T^{OLS}(\frac{debt}{GDP})_{t-1} + \gamma_T^{OLS}(\frac{s}{GDP})_t + \delta_T^{OLS}\Delta(\frac{debt}{GDP})_{t-1} + \\ + \nu_T^{OLS}\Delta(\frac{s}{GDP})_{t-1} + \mu_T^{OLS}\Delta(\frac{s}{GDP})_{t-2} + \epsilon_t\end{aligned} \tag{34}$$

where we added two lagged difference terms of the primary surplus/GDP ratio to obtain white noise residuals. In this case the probability of getting the wrong answer of debt unsustainability is be equal to 0.163, much lower than before. We conclude that one way to disentangle between persistence and sustainability of debt is to consider the evolution of primary surpluses.

9 Some Stylized Facts

The idea that fiscal variables depend on agents' expectations is consistent with some stylized facts. Figure 18 shows the government deficit/GDP ratio (dashed line), the growth rate of output (dotted line) and the two-periods-lagged Consumer Sentiment Index (CCI), in deviation from its sample mean, (solid line) for the US economy. Data are quarterly and refer to the period 1960-2007. Periods of optimism can be identified as those in which the deviation of the CCI from its sample mean is positive. There is a strong negative correlation

[20] The result is robust to a number of alternative specifications: adding lagged difference terms of the dependent variable the probability would be equal to 0.77; including a time trend would lower the probability to 0.652.

between the fiscal deficit/GDP ratio and the level of optimism: roughly, while the growth rate of the economy does not show any substantial variation, during phases of optimism the deficit/GDP ratio decreases, to increase during phases of pessimism.

In Europe as well a similar evidence holds. Table 1 shows for each country the correlation between the deficit/GDP ratio at t and the level of optimism at $t-j$, each column referring to a different j.[21] Data cover the period 1985-2006. Two observations are important. First, Apart for Czech Republic, Hungary and Spain, the correlation between the current value of the primary deficit/GDP ratio and the j-periods-lagged Consumer Confidence Index is negative, implying that the more optimistic were agents' j periods ahead, the lower the current deficit/GDP ratio. Second, in general the correlation decreases with the time distance.

In this section we want to test our theory empirically. In particular, we want to check if, in the data, the government debt responds to agents' expectations in a way that when agents are pessimistic (optimistic) the government issues more (less) debt (and lowers (increases) distortionary taxes) to finance its expenditure, in order to induce agents to revise their expectations upwards (downwards). In the model outlined in section 3, debt at time t depends on the contemporaneous government expenditure shock, on agents' expectations formed at $t-1$ over the marginal utility of consumption at time t, and on the history of all past forecast errors up to $t-1$.

While data on public consumption are easily available, there is no direct series for the history of forecast errors. In order to construct this variable, we proceed in the following way. We use quarterly data contained in the Survey of Professional Forecasters to have the mean one quarter forecast of the Gross Domestic Product.[22] The forecast error in each quarter is defined as the difference between the actual Gross Domestic Product realised at the survey date and the Gross Domestic Product forecast for the same period, normalised by the actual value.[23] In order to work with stationary series, we use the total federal government debt/GDP ratio, and the government expenditure/GDP ratio. Government expenditure includes Federal Consumption Expenditures and Gross Investment. Data are quarterly and cover the period 1980 : 1-2008 : 1 for the U.S. economy.

[21] The Consumer Confidence Index database for Europe can be freely downloaded from the website $http://ec.europa.eu/economy_finance/db_indicators/surveys9185_en.htm$

[22] Data can be freely downloaded from the webpage http://www.philadelphiafed.org/research-and-data/real-time-center/survey-of-professional-forecasters/

[23] Before 1992 agents had to forecast the Gross National Product and after 1992 the Gross Domestic Product. The forecast error has been computed taking this difference into account.

We estimate the following equation

$$\Delta log(\frac{D}{Y})_t = \alpha_0 + \alpha_1 log(\frac{D}{Y})_{t-1} + \alpha_2\gamma_t + \alpha_3 log(\frac{G}{Y})_t + \alpha_4\nu_{t-1} + \alpha_5 log(A_{t-1}) + \epsilon_t \tag{35}$$

where γ_t is the growth rate of real GDP from period $t-1$ till t, $(\frac{D}{Y})_t$ and $(\frac{G}{Y})_t$ are the face value of the debt/GDP ratio and the government expenditure/GDP ratio respectively. ν_t is the forecast error and $A_t = \sum_{k=1980:1}^{t} \nu_k, \forall t = 1980:1, 2008:1$. By construction, a positive forecast error means that agents in the last period were in some sense pessimistic, since they expected that the economy would have performed worse than it actually did. In terms of the model above, $\nu_t > 0$ is equivalent to $\gamma_{t-1} > u_{c,t}$. Table 6 summarises the estimation results. Since the forecast errors are highly correlated, we use the Newey-West standard error estimation. α_1, α_2 and α_3 are statistically significant and their sign is in line with evidence of market incompleteness: an higher government expenditure, or an higher inherited debt, increases the current debt. More importantly, α_4 and α_5 are statistically significant too, and their sign in coherent with the analysis carried over in the paper. In particular, the higher the pessimism in $t-1$, or the more pessimistic have been agents in the past, the higher the debt in t.[24] This result seems to indicate that in U.S. fiscal authorities follow an expectation-dependent plan in a way similar to the one proposed in this paper.

10 Conclusions

Lucas and Stokey (1983) model prescribes to set smooth taxes along the business cycle and to hold sizable state-contingent bond positions. This debt management generates large wealth variation that in turns magnifies the effect of distorted beliefs. As a consequence, small deviations from rational expectations translate into sub-optimal volatility of consumption. Reformulating the government problem to take into account how agents form their expectations leads to very different conclusions in terms of the optimal fiscal policy plan.

There are two main results. The first one is that the policymaker should manipulate agents' beliefs by setting low (high) taxes in a context of pessimism (optimism). This conclusion seems to be supported by the data, and in line with some recent suggestions to handle the up-to-date distress. Moreover, because of the role played by the initial beliefs, this model can account for the wide dispersion across countries in the level of government debt and tax rate, as far as one is willing to accept that different countries have different initial beliefs.

[24] The results are robust to the measure of the forecast error: similar conclusions can be drawn if the forecast error is defined as the difference between actual GDP and one-year ahead expected GDP.

The second one is that the complete market solution under learning is history-dependent. This fact makes assessing market completeness more challenging, since unit-root test in the tax rate can mix evidence of non rational expectations with evidence of market incompleteness. In line with Marcet and Scott (2008) we find that looking at the behaviour of debt is a much more reliable way to test the bond market structure. Also gauging debt sustainability is more complicated because of the persistence induced by agents' expectations.

Several important issues are still open question. First, for simplicity we restrict the analysis to one-period state-contingent bonds. It may be interesting to extend the analysis to the incomplete markets framework where the government can issue bonds at different maturities. Second, we suppose that while agents do not know how aggregate variables are determined, the government has full information about the structure of the economy. Other authors (*inter-alia*, Primiceri (2005) and Cogley and Sargent (2005)) followed the opposite approach. It would be interesting to analyse the case in which neither the households know the government policy rules nor the government knows the households' response to these rules. Third, as in several papers on optimal taxation we abstract from monetary issues. On the other hand, the literature studying the impact of learning on the monetary policy design abstracts from fiscal policy considerations, such as distortionary taxes. A natural step would be to unify these two strands and to understand how the interaction of fiscal and monetary policy can help agents to form their expectations. We leave these issues to a future exercise.

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A Appendix

A.1 A Simple Endowment Economy

We analyse here a non-stochastic endowment economy in which households do not have fully rational expectations. The purpose of the exercise is to identify the fiscal policy design which implements the first-best allocation given non-rational expectations.[25]

In the literature there are two approaches to modelling the behaviour of agents endowed with non-rational expectations: the Euler Equation Approach (henceforth EEA) and the Infinite Horizon Approach (henceforth IHA). In the first, households' decisions depend on one-period-ahead forecasts of future variables appearing in the Euler Equation, while in the second approach they depend on the forecasts regarding outcomes arbitrarily far in the future, insofar as these forecasts satisfy standard probability laws. Without entering into the discussion of which of the two approaches explains better the behaviour of not fully-informed agents, in this section we use both of them to model agents' behaviour and we show that the fiscal policy design is independent of the approach used.

A.2 Modelling learning according to the Euler Equation Approach

The representative household maximises its lifetime utility

$$\widetilde{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{36}$$

subject to the flow budget constraint

$$b_{t-1} + y - T_t = c_t + p_t^b b_t \tag{37}$$

where $y \equiv 1 - g$ is the exogenous and constant output, b_t is the amount of bonds issued by the government at t at the price p_t^b, and T_t is the lump-sum tax in period t.

The Euler equation for consumption is

$$p_t^b = \beta \frac{u_{c,t+1}}{u_{c,t}} \tag{38}$$

Since taxes are lump-sum, the Ricardian Equivalence holds, and there are infinite ways, equivalent in terms of allocations, of financing the public expenditure shock. Among these, we focus on the following policy rules:

[25] Since the results do not change once uncertainty is introduced, we focus only on the non-stochastic framework for clarity.

$$
\begin{aligned}
b_t &= b \forall t &\quad (39)\\
T_t &= T \forall t &\quad (40)
\end{aligned}
$$

The equilibrium in the goods market requires consumption to be equal to output, $c_t = y$. Since agents do not know either that the resource constraint has to hold in any period (since this is one equilibrium condition they have to learn) or that agents are identical, their expectations about consumption tomorrow do not coincide with those that fully-informed agents would have. It follows that the interest rate is not equal to the interest rate under rational expectations, β^{-1}. The question we pose is: given the quantity of bonds issued by the government under rational expectations, what is the fiscal policy in terms of lump-sum taxes which induces agents to consume the exact amount of goods implied by the resource constraint? The following proposition answers the question:

Proposition A.2.1. *Assume that the government fixes the amount of bonds at the RE level, $b^{RE} > 0$. Then the fiscal policy has to be expansionary (restrictive) whenever agents' expectations over one-step-ahead marginal utility of consumption are higher than the expectations fully rational agents would have.*

Proof. The proof is immediate. Under RE, combining the flow budget constraint with the resource constraint we get

$$
T^{RE} = (1 - \beta)b^{RE} \quad (41)
$$

Let $\gamma_t \equiv \widetilde{E}_t u_{c,t+1}$. Combining equation (37) after substituting for the agents' optimality condition with the resource constraint we get

$$
T_t^L = (1 - \beta \frac{\gamma_t}{u_c}) b^{RE} \quad (42)
$$

Subtracting equation (42) from (41) we obtain

$$
T^{RE} - T_t^L = b^{RE} \beta (\frac{\gamma_t}{u_c} - 1) \quad (43)
$$

Insofar as agents are pessimistic, i.e. $\gamma_t > u_c$, $T^{RE} > T_t^L$. Symmetrically, when agents are optimistic, i.e. $\gamma_t < u_c$, $T^{RE} < T_t^L$.

□

If the belief over the next period marginal utility coincides with the one under rational expectations, the equilibrium tax under learning would also coincide with that prevailing under rational expectations. For pessimistic (optimistic) agents, i.e. for agents expecting their marginal utility to be higher (lower) than it will actually be, the government has to reduce (increase) taxes relative to the RE benchmark. The reason is the following. Pessimistic agents expect future taxes to be higher than today, and therefore to smooth consumption they want to save more and consume less today: an expansionary fiscal policy is required to clear the goods market. The further away the initial belief is compared with that under rational expectations, the higher is the difference between taxes under rational expectations and under learning.

A.3 Modelling learning according to the Infinite Horizon Approach

In subsection A.2, only one-period-ahead expectations over marginal utility matter in order to pin down agents' optimal choice. In a series of papers (see Preston (2005a) and Preston (2005b) among others) Preston shows two possible shortcomings intrinsic to the EEA approach. The first is that the expectation of next period consumption based on the regression of past observations on aggregate shocks does not necessarily coincide with the one implied by forwarding the optimal consumption rule, unless agents have already converged to the rational expectations equilibrium; in other words, the EEA is not micro founded. The second problem is that agents do not take into account their wealth, and therefore they would violate ex ante their transversality condition.

In this subsection we show that, even when agents make long-run forecasts about future macroeconomic conditions, the tax policy which ensures the goods market clearing shares the same qualitative features as the one highlighted in the previous subsection.

As before, households' problem is to maximise equation (36) subject to (37). Using forward substitution in the flow budget constraint (37), using (38) and imposing the transversality condition

$$\lim_{T\to\infty}\left(\prod_{k=0}^{T}\frac{1}{r_{t+k}}\right)b_{t+T}=0 \tag{44}$$

we get

$$b_{t-1}=\frac{1}{1-\beta}c_t-y+T_t-\sum_{j=1}^{\infty}\left(\prod_{k=0}^{j-1}\frac{1}{r^e_{t+k}}\right)(y-T^e_{t+j}) \tag{45}$$

where r^e_{t+k} and T^e_{t+j} denote agents' expectations regarding the interest rate prevailing from $t+k$ to $t+k+1$ and the tax at $t+j$.

Let $r^e_{t+k} \equiv \tilde{R}$ and $T^e_{t+j} \equiv \tilde{T}$. The optimal consumption rule is therefore given by

$$c_t = (1-\beta)\{b_{t-1} + y - T_t + \frac{1}{r_t}\{\frac{\tilde{R}}{\tilde{R}-1}\}(y-\tilde{T})\} \tag{46}$$

Optimal consumption is an increasing function of the current wealth, b_{t-1} and a decreasing function of both current and expected interest rates and of current and expected endowments net of taxes. As before, we want to characterise the fiscal policy which ensures equilibrium in the goods market, assuming that the bond policy function is given by equation (39). The next proposition, similar in spirit to proposition A.2.1, answers the question.

Proposition A.3.1. *Assume that the government fixes the amount of bonds at the RE level, $b^{RE} > 0$, and that $\tilde{R} > 1$. Then the higher are the expected tax liabilities, the lower is the current tax rate the government has to set.*

Proof. Forwarding equation (46) the expected one-period-ahead consumption is

$$\tilde{E}_t c_{t+1} = (1-\beta)(b + y - \tilde{T} + \frac{1}{\tilde{R}}\frac{\tilde{R}}{\tilde{R}-1}(y-\tilde{T}) \tag{47}$$

Assuming a logarithmic utility function the interest rate is given by

$$\frac{1}{r_t} = \beta\frac{c_t}{(1-\beta)(b + y - \tilde{T} + \frac{1}{\tilde{R}}\frac{\tilde{R}}{\tilde{R}-1}(y-\tilde{T})} \tag{48}$$

Inserting (48) into equation (46) we get

$$c_t = \frac{b + (y-\tilde{T})\frac{\tilde{R}}{\tilde{R}-1}}{b + (y-\tilde{T})\frac{\tilde{R}}{\tilde{R}-1}(1-\beta)}(1-\beta)(b + y - T_t) \tag{49}$$

It is immediate to show that if $\tilde{T} = (1-\beta)b$ and $\tilde{R} = \frac{1}{\beta}$, which is the REE, then automatically $c_t = y \forall t$. Rearranging terms in equation (49) we express the current tax liabilities as a function of agents expectations' about future taxes and interest rates,

$$T_t = b + y - \frac{\left(b + (y-\tilde{T})\frac{\tilde{R}}{\tilde{R}-1}(1-\beta)\right)y}{(1-\beta)\left(b + (y-\tilde{T})\frac{\tilde{R}}{\tilde{R}-1}\right)} \tag{50}$$

Taking the derivatives of equation (50) with respect to $\tilde{T}$, after some algebra we get

$$\frac{\partial T_t}{\partial \tilde{T}} = -\frac{\tilde{R}}{\tilde{R}-1}(1-\beta)by < 0 \tag{51}$$

□

A.4 Proof of Proposition 2.2

Let the perceived law of motion be equal to

$$\hat{u}_{c,t} = d_{t-1}\hat{g}_{t-1} + \hat{\epsilon}_t \tag{52}$$

Substituting prices from the households' optimality conditions into the period by period budget constraint and loglinearizing around the rational expectations equilibrium we get

$$\bar{b}u_{\bar{c}}\hat{b}_{t-1} + \bar{b}u_{\bar{c}\bar{c}}\bar{c}\hat{c}_t = (u_{\bar{c}\bar{c}}\bar{c} + u_{\bar{c}})\bar{c}\hat{c}_t - (u_{\bar{l}\bar{l}}(1-\bar{l}) - u_{\bar{l}})\bar{l}\hat{l}_t + \beta\bar{b}u_{\bar{c}}\tilde{E}_t\hat{u}_{c,t+1} + \beta\bar{b}u_{\bar{c}}E_t\hat{b}_t \tag{53}$$

where $\bar{x} = x^{stst}$.

Substituting for the bond policy function under rational expectations and for the resource constraint we get

$$\hat{c}_t = \frac{[u_{\bar{l}\bar{l}}(1-\bar{l}) - u_{\bar{l}}]\bar{g}\hat{g}_t - \bar{b}u_{\bar{c}}\psi^b_{RE}(1-\beta\rho_g)\hat{g}_t + \beta\bar{b}u_{\bar{c}}d_{t-1}\hat{g}_t}{\bar{c}[\bar{b}u_{\bar{c}c} - (u_{\bar{c}\bar{c}}\bar{c} + u_{\bar{c}}) + u_{\bar{l}} - u_{\bar{l}\bar{l}}(1-\bar{l})]} \tag{54}$$

from which follows that

$$\frac{\partial\hat{u}_{c,t}}{\partial d_{t-1}} = \frac{-\beta\bar{b}u_{\bar{c}}\hat{g}_t}{\bar{c}[\bar{b}u_{\bar{c}c} - (u_{\bar{c}\bar{c}}\bar{c} + u_{\bar{c}}) + u_{\bar{l}} - u_{\bar{l}\bar{l}}(1-\bar{l})]} \tag{55}$$

being $\hat{u}_{c,t} = -\hat{c}_t$. Assuming $u_{c,t} = log(c_t) + l_t$, the right side of equation (55) becomes $\frac{-\beta\bar{b}}{-\bar{b}+\bar{c}^2}$. The E-stability condition is satisfied iff $\frac{-\beta\bar{b}}{-\bar{b}+\bar{c}^2} < 1$. The intertemporal budget constraint implies that in steady state it has to be the case that $\bar{b}(1-\beta) = \bar{c}\bar{l}$. Hence using the resource constraint the learnability condition becomes

$$c > \frac{1-\bar{g}}{2} \tag{56}$$

Because of the assumed utility function, we can show that the inequality in (56) is always satisfied. Using the government FOC with respect to consumption we get that $c_t = \frac{1}{1+\Delta}$, where Δ is the Lagrange multiplier associated with the implementability condition $E_0\sum_{t=0}^{\infty}\beta^t(u_{c,t}c_t - u_{l,t}(1-l_t) = b_{-1}u_{c,0}$. $E_0\sum_{t=0}^{\infty}\beta^t l_t = E_0\sum_{t=0}^{\infty}\beta^t\frac{\Delta-(1+\Delta)g_t}{1+\Delta} = 0$, where the first equality holds for the specific utility function and the second one holds assuming that $b_{-1} = 0$. If $g_0 = \bar{g}$, $E_0\sum_{t=0}^{\infty}\beta^t g_t = \frac{\bar{g}}{1-\beta}$, from which $\Delta = \frac{\bar{g}}{1-\bar{g}}$. Inserting into optimal consumption under rational expectations we get that $c_t = (1-\bar{g}) > \frac{1-\bar{g}}{2}$.

A.5 Proof of Proposition 3.0.2

First we show that constraints (3), (13) and (14) imply (19).

Consider the period-by-period budget constraint after substituting for the household optimality conditions:

$$b_{t-1}(g_t) = \frac{u_{c,t}(g_t)s_t(g_t)}{u_{c,t}(g_t)} + \beta\frac{\gamma_t^H b_t(H)}{u_{c,t}(g_t)}\pi(H) + \beta\frac{\gamma_t^L b_t(L)}{u_{c,t}(g_t)}\pi(L) \tag{57}$$

where $s_t \equiv c_t - \frac{u_{l,t}}{u_{c,t}}(1-l_t)$, $b_t(i)$ for $i = H, L$ is the amount of state-contingent bond holdings, and γ_t^i for $i = H, L$ are the (state contingent) marginal utilities that agents expect in the next period and $\pi(i) = \pi(g_{t+1} = g^i|g_t)$ for $i = H, L$. Since the fiscal authority has full information about the economy, in t it will issue state-contingent bonds such that the budget constraint in the next period is satisfied for any realisation of the government shock. Forwarding (57) one period we get:

$$b_t(H) = \frac{u_{c,t+1}(H)s_{t+1}(H)}{u_{c,t+1}(H)} + \beta\frac{\gamma_{t+1}^H b_{t+1}(H)}{u_{c,t+1}(H)}\pi(H) + \beta\frac{\gamma_{t+1}^L b_{t+1}(L)}{u_{c,t+1}(H)}\pi(L) \tag{58}$$

$$b_t(L) = \frac{u_{c,t+1}(L)s_{t+1}(L)}{u_{c,t+1}(L)} + \beta\frac{\gamma_{t+1}^H b_{t+1}(H)}{u_{c,t+1}(L)}\pi(H) + \beta\frac{\gamma_{t+1}^L b_{t+1}(L)}{u_{c,t+1}(L)}\pi(L) \tag{59}$$

Substituting (58) and (59) into (57), and multiplying both sides by $x_{t-1} \equiv [\gamma_{t-1}^H I(g_t = g^H) + \gamma_{t-1}^L I(g_t = g^L)]$

$$\begin{aligned} b_{t-1}(g_t)x_{t-1} = u_{c,t}s_t\frac{x_{t-1}}{u_{c,t}} + \frac{x_{t-1}}{u_{c,t}}\beta\{&\frac{\gamma_t^H}{u_{c,t+1}(H)}[u_{c,t+1}(H)s_{t+1}(H)+ \\ &\beta\gamma_{t+1}^H b_{t+1}(H)\pi(H) + \beta\gamma_{t+1}^L b_{t+1}(L)\pi(L)]\pi(g_{t+1} = g^H|g_t) + \frac{\gamma_t^L}{u_{c,t+1}(L)}[u_{c,t+1}(L)s_{t+1}(L)+ \\ &\beta\gamma_{t+1}^H b_{t+1}(H)\pi(H) + \beta\gamma_{t+1}^L b_{t+1}(L)\pi(L)]\pi(g_{t+1} = g^L|g_t)\} \end{aligned} \tag{60}$$

Define $W_t = \frac{x_{t-1}}{u_{c,t}}$. Keeping substituting forward we get

$$b_{t-1}(g_t)x_{t-1} = E_t\sum_{j=0}^{\infty}\beta^j\prod_{k=t}^{t+j}W_k u_{c,t+j}s_{t+j} \tag{61}$$

Define

$$\tilde{A}_{t+j} \equiv \prod_{k=t}^{t+j}W_k \tag{62}$$

and multiply each side of (61) by $H_{t-1} \equiv \prod_{k=0}^{t-1} W_k$. We get

$$b_{t-1}(g_t)x_{t-1}H_{t-1} = E_t \sum_{j=0}^{\infty} \beta^j A_{t+j} u_{c,t+j} s_{t+j} \tag{63}$$

where $A_{t+j} = H_{t-1} \prod_{k=t}^{t+j} W_k$.
Notice that the A_{t+j} has a recursive formulation given by:

$$\begin{aligned} A_{t+j} &= \prod_{k=0}^{t-1} W_k \times \prod_{k=t}^{t+j} W_k = \prod_{k=0}^{t+j} W_k = \prod_{k=0}^{t+j-1} W_k \times \frac{x_{t+j-1}}{u_{c,t+j}} = \\ &= A_{t+j-1} \frac{x_{t+j-1}}{u_{c,t+j}} \end{aligned} \tag{64}$$

It follows that

$$A_t \equiv H_t = H_{t-1} \frac{x_{t-1}}{u_{c,t}} \tag{65}$$

To prove the reverse implication, take any feasible allocation $\{c_{t+j}, l_{t+j}\}_{j=0}^{\infty}$ that satisfies equation (19). Then it is always possible to back out the state-contingent bond holding such that the period-by-period budget constraint is satisfied.

From equation (63), define

$$b_{t-1}(g_t) = E_t \sum_{j=0}^{\infty} \beta^j A_{t+j} u_{c,t+j} s_{t+j} \frac{1}{x_{t-1} H_{t-1}} \tag{66}$$

It follows that

$$b_t(g_{t+1}) = E_{t+1} \sum_{j=0}^{\infty} \beta^j A_{t+1+j} u_{c,t+1+j} s_{t+1+j} \frac{1}{x_t H_t} \tag{67}$$

$$
\begin{aligned}
b_{t-1}(g_t) &= \frac{A_t u_{c,t} s_t}{x_{t-1}H_{t-1}} + E_t \sum_{j=1}^{\infty} \beta^j A_{t+j} u_{c,t+j} s_{t+j} \frac{1}{x_{t-1}H_{t-1}} = \\
&= \frac{A_t u_{c,t} s_t}{x_{t-1}H_{t-1}} + \beta E_t \sum_{j=0}^{\infty} \beta^j A_{t+1+j} u_{c,t+1+j} s_{t+1+j} \frac{1}{x_{t-1}H_{t-1}} = \\
&= \frac{A_t u_{c,t} s_t}{x_{t-1}H_{t-1}} + \frac{\beta}{x_{t-1}H_{t-1}} E_t\{x_t H_t [\frac{\sum_{j=0}^{\infty} \beta^j A_{t+1+j} u_{c,t+1+j} s_{t+1+j}}{x_t H_t}]\} = \\
&= \frac{A_t u_{c,t} s_t}{x_{t-1}H_{t-1}} + \frac{\beta}{x_{t-1}H_{t-1}} E_t\{x_t H_t [E_{t+1} \frac{\sum_{j=0}^{\infty} \beta^j A_{t+1+j} u_{c,t+1+j} s_{t+1+j}}{x_t H_t}]\} = \\
&= \frac{A_t u_{c,t} s_t}{x_{t-1}H_{t-1}} + \frac{\beta}{x_{t-1}H_{t-1}} E_t\{x_t H_t B_t\} =
\end{aligned}
\tag{68}
$$

Using (65) we get

$$
\begin{aligned}
b_{t-1}(g_t) &= s_t + \frac{\beta}{u_{c,t}} E_t([\gamma_t^H I(g_{t+1} = g^H) + \gamma_t^L I(g_{t+1} = g^L)] b_t(g_{t+1})) = \\
&= s_t + \frac{\beta}{u_{c,t}} [\gamma_t^H b_t(g_{t+1} = g^H) \pi(g_{t+1} = g^H | g_t) + \gamma_t^L b_t(g_{t+1} = g^L) \pi(g_{t+1} = g^L | g_t)]
\end{aligned}
\tag{69}
$$

A.6

Attach the multipliers Δ, $\beta^t \pi_t(g^t)\lambda_{1,t}(g^t)$, $\beta^t \pi_t(g^t)\lambda_{2,t}(g^t)$, $\beta^t \pi_t(g^t)\lambda_{3,t}(g^t)$ and $\beta^t \pi_t(g^t)\lambda_{4,t}(g^t)$ to constraints (20), (22) for $i = H, L$, (23) and to (21).

The Lagrangian is

$$
\begin{aligned}
\mathcal{L} =& E_0 \sum_{t=0}^{\infty} \beta^t \{u(c_t, l_t) + \Delta(A_t(u_{c,t} c_t - u_{l,t}(1 - l_t))) \\
&+ \lambda_{1,t}((\gamma_t^H - \gamma_{t-1}^H) I(g_t = g_L) + (\gamma_t^H - (1 - \alpha_t)\gamma_{t-1}^H - \alpha_t u_{c,t}) I(g_t = g_H)) \\
&+ \lambda_{2,t}((\gamma_t^L - \gamma_{t-1}^L) I(g_t = g_H) + (\gamma_t^L - (1 - \alpha_t)\gamma_{t-1}^L - \alpha_t u_{c,t}) I(g_t = g_L)) \\
&+ \lambda_{3,t}(1 - l_t - c_t - g_t)\} + \lambda_{4,t}(A_t - A_{t-1} \frac{\gamma_{t-1}^H I(g_t = g^H) + \gamma_{t-1}^L I(g_t = g^L)}{u_{c,t}}) - \Delta A_0 u_{c,0} b_{-1}
\end{aligned}
$$

Assuming $b_{-1} = 0$, the first-order necessary conditions $\forall t \geq 0$ are:

- c_t:

$$\begin{aligned} &u_{c,t} + \Delta A_t(u_{cc,t}c_t + u_{c,t}) - \lambda_{1,t}\alpha_t u_{cc,t} I(g_t = g^H) \\ &- \lambda_{2,t}\alpha_t u_{cc,t} I(g_t = g^L) + u_{cc,t}\lambda_{4,t}A_{t-1}\frac{\gamma_{t-1}^H I(g_t = g^H) + \gamma_{t-1}^L I(g_t = g^L)}{u_{c,t}^2} = \lambda_{3,t} \end{aligned} \tag{70}$$

- l_t:

$$u_{l,t} + \Delta A_t(u_{l,t} - u_{ll,t}(1 - l_t)) = \lambda_{3,t} \tag{71}$$

- γ_t^H:

$$\begin{aligned} &\lambda_{1,t} - \beta E_t\{\lambda_{1,t+1}I(g_{t+1} = g^L) + (1 - \alpha_{t+1})\lambda_{1,t+1}I(g_{t+1} = g^H) + \\ &+ \frac{\lambda_{4,t+1}A_t}{u_{c,t+1}}I(g_{t+1} = g^H)\} = 0 \end{aligned} \tag{72}$$

- γ_t^L:

$$\begin{aligned} &\lambda_{2,t} - \beta E_t\{\lambda_{2,t+1}I(g_{t+1} = g^H) + (1 - \alpha_{t+1})\lambda_{2,t+1}I(g_{t+1} = g^L) + \\ &+ \frac{\lambda_{4,t+1}A_t}{u_{c,t+1}}I(g_{t+1} = g^L)\} = 0 \end{aligned} \tag{73}$$

- A_t:

$$\Delta(u_{c,t}c_t - u_{l,t}(1 - l_t)) + \lambda_{4,t} - \beta E_t\lambda_{4,t+1}\frac{\gamma_t^H I(g_{t+1} = g^H) + \gamma_t^L I(g_{t+1} = g^L)}{u_{c,t+1}} \tag{74}$$

From equation (74)

$$\lambda_{4,t} = -\Delta(u_{c,t}c_t - u_{l,t}(1 - l_t)) + \beta E_t\lambda_{4,t+1}\frac{\gamma_t^H I(g_{t+1} = g^H) + \gamma_t^L I(g_{t+1} = g^L)}{u_{c,t+1}} \tag{75}$$

Multiplying both sides by A_t we get

$$\begin{aligned} \lambda_{4,t}A_t &= -\Delta A_t(u_{c,t}c_t - u_{l,t}(1 - l_t)) + \beta E_t\lambda_{4,t+1}\frac{A_t\gamma_t^H I(g_{t+1} = g^H) + \gamma_t^L I(g_{t+1} = g^L)}{u_{c,t+1}} = \\ &= -\Delta A_t(u_{c,t}c_t - u_{l,t}(1 - l_t)) + \beta E_t\lambda_{4,t+1}A_{t+1} \end{aligned} \tag{76}$$

where the last equality follows from equation (21).

Iterating forward we obtain

$$\lambda_{4,t}A_t = -\Delta E_t \sum_{j=0}^{\infty} \beta^j A_{t+j}(u_{c,t+j}c_{t+j} - u_{l,t+j}(1-l_{t+j})) \tag{77}$$

Inserting (77) into (70) we get

$$\begin{aligned} &u_{c,t} + \Delta A_t(u_{cc,t}c_t + u_{c,t}) - \lambda_{1,t}\alpha_t u_{cc,t} I(g_t = g^H) \\ &- \lambda_{2,t}\alpha_t u_{cc,t} I(g_t = g^L) - \Delta \frac{u_{cc,t}}{u_{c,t}} E_t \sum_{j=0}^{\infty} \beta^j A_{t+j}(u_{c,t+j}c_{t+j} - u_{l,t+j}(1-l_{t+j})) = \lambda_{3,t} \end{aligned} \tag{78}$$

A.7

Under rational expectations the following equalities hold

$$\begin{aligned} \gamma_{t-1}^H &= u_{c,t}(g_t = g^H)\forall t \\ \gamma_{t-1}^L &= u_{c,t}(g_t = g^L)\forall t \end{aligned}$$

which implies that $A_t = A_{t-1} = 1\forall t$. The Lagrangian collapses to

$$\begin{aligned} \mathcal{L} =& E_0 \sum_{t=0}^{\infty} \beta^t [u(c_t, l_t) + \Delta(u_{c,t}c_t - u_{l,t}(1-l_t)) + \lambda_{1,t}(\gamma_t^H - \gamma_{t-1}^H) + \lambda_{2,t}(\gamma_t^L - \gamma_{t-1}^L) \\ &+ \lambda_{3,t}(1 - l_t - c_t - g_t)] - \Delta u_{c,0} b_{-1} \end{aligned}$$

The first-order conditions with respect to γ_t^H and γ_t^L are

- γ_t^H :

$$\lambda_{1,t} = \beta E_t \lambda_{1,t+1} \tag{79}$$

- γ_t^L :

$$\lambda_{2,t} = \beta E_t \lambda_{2,t+1} \tag{80}$$

which imply that the only solution is $\lambda_{1,t} = \lambda_{2,t} = 0$. The first-order condition with respect to consumption and leisure are

$$u_{c,t} + \Delta(u_{cc,t}c_t + u_{c,t}) = \lambda_{3,t} \tag{81}$$

$$u_{l,t} + \Delta(u_{l,t} - u_{ll,t}(1 - l_t)) = \lambda_{3,t} \tag{82}$$

which are exactly the optimality conditions found in a rational expectations framework (see Lucas and Stokey (1983)) in which expectations do not depend on the current consumption level and in which there is no distortion into agents' beliefs that the government has to manipulate optimally.

A.8 Proof of Proposition 4.1.1

The period-by-period budget constraint implies that the following equality

$$b_{t-1}A_{t-1}\gamma_{t-1} = \sum_{j=0}^{\infty} \beta^j A_{t+j}(u_{c,t+j}c_{t+j} - u_{l,t+j}(1 - l_{t+j})) \tag{83}$$

Equation (28) can be written as

$$\begin{aligned} &u_{c,t} + \Delta A_t(u_{cc,t}c_t + u_{c,t}) - \lambda_{1,t}\alpha_t u_{cc,t} - \\ &\Delta \frac{u_{cc,t}}{u_{c,t}} b_{t-1}A_{t-1}\gamma_{t-1} = u_{l,t} + \Delta A_t(u_{l,t} - u_{ll,t}(1 - l_t)) \end{aligned} \tag{84}$$

The optimal policy under rational expectations is to run a balanced budget. Assuming that agents' initial belief over marginal utility of consumption is close to the actual marginal utility under rational expectations, $A_{t-1} \approx 1$. Moreover, beliefs close to rational expectations implies that also the optimal fiscal policy under learning is close to that under rational expectations, therefore $b_{t-1} \approx 0$. Under this assumption and the one that α_t is small enough, equation (28) can be rewritten as

$$u_{c,t} + \Delta A_t(u_{cc,t}c_t + u_{c,t}) \approx u_{l,t} + \Delta A_t(u_{l,t} - u_{ll,t}(1 - l_t)) \tag{85}$$

Using equation (29) we get that

$$u_{c,t} = \frac{2\Delta\gamma_{t-1}A_{t-1}}{-1 + \sqrt{1 + 4\Delta A_{t-1}\gamma_{t-1}}} \tag{86}$$

1. Taking the first derivative of (86) we get

$$\frac{\partial u_{c,t}}{\partial \gamma_{t-1}} = \frac{-1 + \sqrt{1 + 4\Delta\gamma_{t-1}} - 2\Delta\gamma_{t-1}\frac{1}{\sqrt{1+4\Delta\gamma_{t-1}}}}{(-1 + \sqrt{1 + 4\Delta\gamma_{t-1}})^2} \tag{87}$$

which is positive being $\Delta > 0$ and $\gamma_{t-1} > 0$.

The second derivative with respect to γ_{t-1} is equal to

$$\frac{\partial^2 u_{c,t}}{\partial \gamma_{t-1}^2} = \frac{\frac{4\Delta^2\gamma_{t-1}}{\sqrt{x}x}(-1+\sqrt{x})^2 - \frac{4\Delta}{\sqrt{x}}(-1+\sqrt{x})(-1+\sqrt{x}-\frac{2\Delta\gamma_{t-1}}{\sqrt{x}})}{(-1+\sqrt{x})^4} \tag{88}$$

where $x = \sqrt{1+4\Delta\gamma_{t-1}}$. After some algebra, it can be shown that equation (88) is negative, being $\Delta > 0$ and $\gamma_{t-1} > 0$

2. Imposing $T(\gamma^*, A_{t-1}) = \gamma^*$ we get that the fixed point is given by

$$\gamma^* = 1 + \Delta A_{t-1} > 0 \tag{89}$$

3. Imposing $\gamma_{t-1} < \frac{3}{4\Delta}$ implies that $-1+\sqrt{1+4\Delta\gamma_{t-1}} < 1$. It follows that the learnability condition

$$\frac{\partial u_{c,t}}{\partial \gamma_{t-1}} |_{\gamma^*} < 1 \tag{90}$$

is satisfied because

$$-\frac{2\Delta\gamma^*}{\sqrt{1+4\Delta A_{t-1}}} < 0 \tag{91}$$

A.9 Proof of Proposition 4.1.2

Suppose that if in period t $u_{c,t} - \gamma_{t-1} < 0$, then $u_{c,t+1} - \gamma_t < 0$ as well. This condition is satisfied for small enough values of α_t.

First notice that consumption under rational expectations is a special case of the previous equation when the agents' belief about today's marginal utility coincides with the actual marginal utility and the product of the past ratios between expected and actual marginal utility is 1.

$$c^{RE} = c^L(u_{c,t}, 1, b_{-1}) \tag{92}$$

Consumption under learning converges to the one under rational expectations if and only if

$$lim_{t\to\infty} log A_t = 0 \tag{93}$$

since under rational expectations $logA_t \equiv 0$. When the government expenditure shock is constant, the law of motion for A_t is given by

$$A_t = A_{t-1}\frac{\gamma_{t-1}}{u_{c,t}} \tag{94}$$

Substituting backwards in the definition of (94) and taking log we get that

$$lim_{t\to\infty} log A_t = lim_{t\to\infty} \sum_{j=0}^{t} log \frac{\gamma_{j-1}}{u_{c,j}} \tag{95}$$

Using the result in proposition 4.1.1 we know that the expected marginal utility converges to the actual one. Define N the time when this happens. Then

$$lim_{t\to\infty} log A_t = \sum_{j=0}^{N} log \frac{\gamma_{j-1}}{u_{c,j}} \tag{96}$$

Being the finite sum of finite numbers, $logA_t$ converges to a strictly positive value for initial pessimistic belief and to a strictly negative value for initial optimistic belief. Since the implementability condition under learning does not converge to the one under rational expectations, the allocation does not either.

To show the second part of the proposition, assume for simplicity the same utility function as in equation (29). Equation (82) implies that

$$c_t = \frac{1}{1+\Delta} \forall t \geq 1 \tag{97}$$

At $t = 0$, for any given b_{-1}, consumption under RE is equal to

$$c_0 = \frac{1 + \sqrt{(}1 + 4\Delta b_{-1}(1+\Delta))}{2(1+\Delta)} \tag{98}$$

Imposing the resource constraint, the implementability condition $\sum_{t=0}^{\infty} \beta^t (u_{c,t}c_t - u_{l,t}(1 - l_t)) = u_{c,0}b_{-1}$ implies that

$$\begin{aligned} l_0 + \sum_{t=1}^{\infty} \beta^t l_t = 1 - c_0 - g + \frac{\beta}{1-\beta}(1 - \frac{1}{1+\Delta} - g) = \\ \frac{1-g}{1-\beta} - \frac{\beta}{1-\beta}\frac{1}{1+\Delta} = b_{-1} \frac{4(1+\Delta)^2 + (1 + \sqrt{(}1 + 4\Delta b_{-1}(1+\Delta))^2}{2(1 + \sqrt{(}1 + 4\Delta b_{-1}(1+\Delta)))(1+\Delta)} \end{aligned} \tag{99}$$

Denote the positive root of equation (99) as $\Delta^* = \Delta(b_{-1})$. Inserting Δ^* into equation (97) we get

$$c_t = \frac{1}{1+\Delta^*} = \frac{1}{1+\Delta(b_{-1})} \tag{100}$$

Let $c^L = lim_{t\to\infty} c_t^L(\gamma_{-1})$. The initial holding of bonds such that the allocation under learning converges to the one under rational expectations starting with that amount of bond is defined by the equation

$$c^L = \frac{1}{1+\Delta(b_{-1})} \tag{101}$$

A.10

To discuss the quality of the learning equations used by agents to predict one-step-ahead state-contingent marginal utility of consumption, we use the Epsilon-Delta Rationality criterion (EDR), as formalized in Marcet and Nicolini (2003). Define

$$\pi^{\epsilon,T} \equiv P(\frac{1}{T}\sum_{t=0}^{T}[u_{c,t}-\gamma_{t-1}]^2 < \frac{1}{T}\sum_{t=0}^{T}[u_{c,t}-E_{t-1}u_{c,t}]^2+\epsilon)$$

which is a function of ϵ, δ and T. $E_{t-1}u_{c,t}$ denote the expectations of an agent who knows the whole economic structure of the model. The learning mechanism (15) with $\alpha_t = \alpha$ satisfies EDR for (ϵ,δ,T) if $\pi^{\epsilon,T} \geq 1-\delta$. Table 7 shows this $\pi^{\epsilon,T}$ for different values of ϵ (across columns) and T (across rows). Reported values are computed out of 1000 simulations. Since the discount factor is equal to 0.95, each period corresponds to one year. After 10 years, there is an 80 percent probability that the prediction error made by boundedly rational agents is at most 3 percent higher than the prediction error made by fully rational agents. We conclude that along the transition agents use a learning scheme that generate quite good forecasts.

Table 1: Correlation between deficit/GDP ratio and j-periods-lagged agents optimism. The first column refers to $j = 1$,the second to $j = 2$, and the third to $j = 3$

Country	$j = 1$	$j = 2$	$j = 3$
Belgium	-0.43	-0.37	-0.28
Czech Republic	0.05	0.04	-0.29
Denmark	-0.55	-0.49	-0.53
Germany	-0.07	-0.44	-0.19
Estonia	-0.6	-0.57	-0.17
Ireland	-0.64	-0.3	0.14
Greece	-0.36	-0.18	0.41
France	-0.63	-0.55	-0.23
Italy	-0.37	-0.36	-0.26
Hungary	0.55	-0.03	-0.15
Netherland	-0.45	-0.69	-0.5
Austria	-0.32	-0.14	-0.24
Portugal	-0.22	-0.4	-0.65
Finland	-0.21	-0.32	-0.42
Spain	0.1	0.02	0.11
Sweden	-0.7	0.08	0.41
UK	-0.8	-0.78	-0.6

Table 2: Statistics of the allocation under rational expectations

	Mean	St.Dev.	Autocorr
consumption	0.423	0.01	0.6
leisure	0.5	0.01	0.6
labor tax rate	0.16	0.003	0.6
market value of debt	0.04	0.03	0.6
primary surplus	0.003	0.02	0.6

Table 3: Statistics under learning after convergence of beliefs

	Initially pessimistic agents			Initially optimistic agents		
	Mean	St.Dev.	Autocorr	Mean	St.Dev.	Autocorr
consumption	.416	.015	.6	.45	1e-4	.6
leisure	.51	.01	.6	.47	1e-4	.6
labor tax rate	.18	.01	.6	.04	3e-6	.6
market value of debt	.38	.04	.6	-0.35	3e-4	.6
primary surplus	.015	.02	.6	-.056	.02	.6

Table 4: Statistics under RE and learning for the first 30 periods

	Rational expectations			Initially pessimistic agents		
	Mean	St.Dev.	Autocorr	Mean	St.Dev.	Autocorr
consumption	.42	.012	.53	.43	.015	.62
leisure	.5	.012	.53	.49	.014	.73
labor tax rate	.16	.003	.526	.14	.03	.95
market value of debt	.04	.03	.526	.28	.08	.86
primary surplus	.001	.02	.526	-.006	.02	.73

Table 5: OLS estimates and t-statistics (in parenthesis) with i.i.d. government shock

	α	β	R^2
$\tau_t^{RE} = \alpha + \beta\tau_{t-1}^{RE} + \varepsilon_t$	0.1562 (7.0467)	-0.0171 (-0.1207)	0.9996
$\tau_t^{L} = \alpha + \beta\tau_{t-1}^{L} + \varepsilon_t$	0.0288 (2.3923)	0.8213 (10.7910)	0.9960

Table 6: Newey-West standard error estimates and t-statistics (in parenthesis)

α_0	α_1	α_2	α_3	α_4	α_5	R^2_{adj}
-0.0 (-0.15)	-0.12 (-6.5)	-1.92 (-5.32)	0.24 (10.97)	0.18 (3.27)	0.16 (6.43)	0.57

Table 7: $\pi^{\epsilon,T}$

$T\backslash\epsilon$	0.04	0.03	0.02
5	1	.4	0
10	1	.8	0
15	1	1	.06
20	1	1	.4

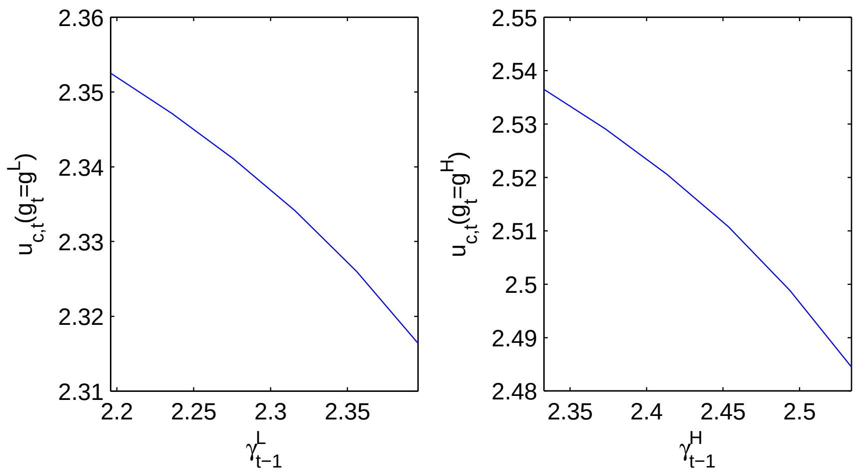

Figure 1: T-mapping

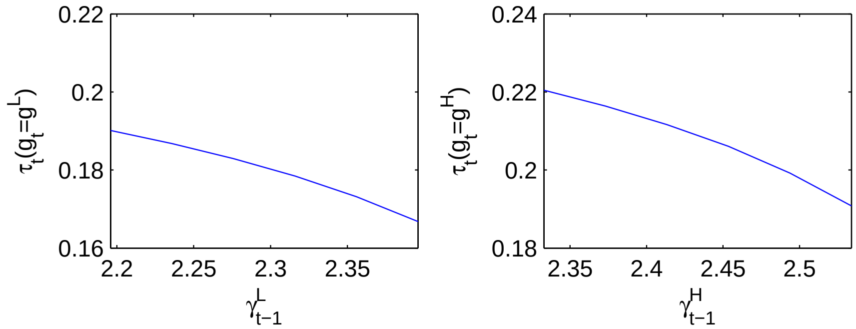

Figure 2: Tax policy function

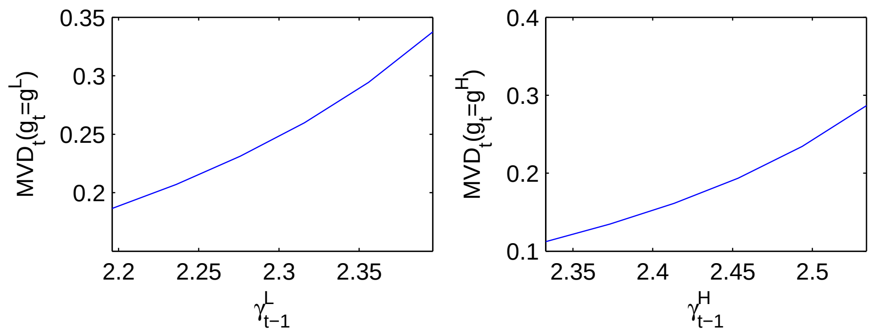

Figure 3: Bond holdings policy function

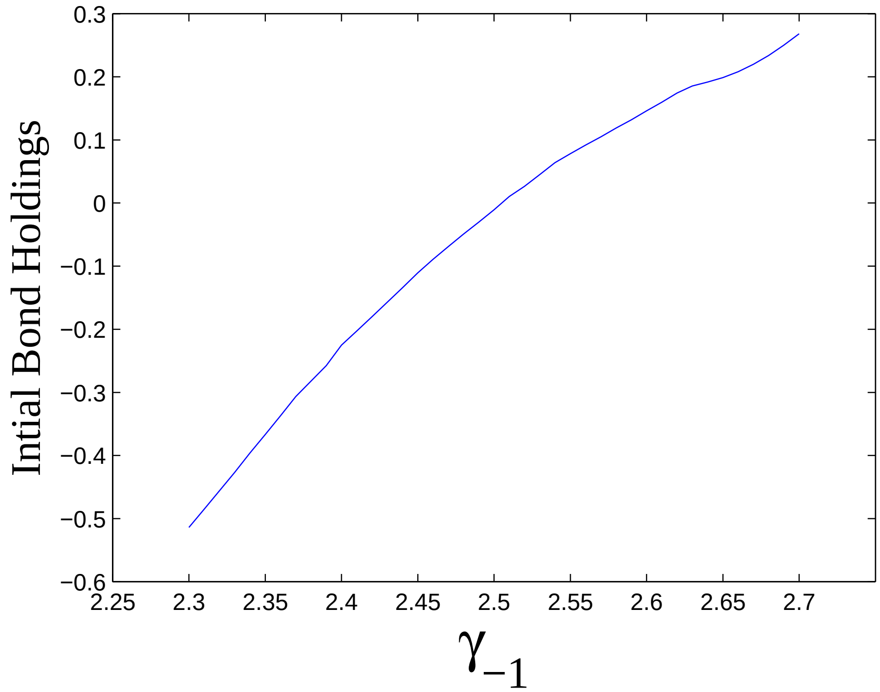

Figure 4: Beliefs and initial debt

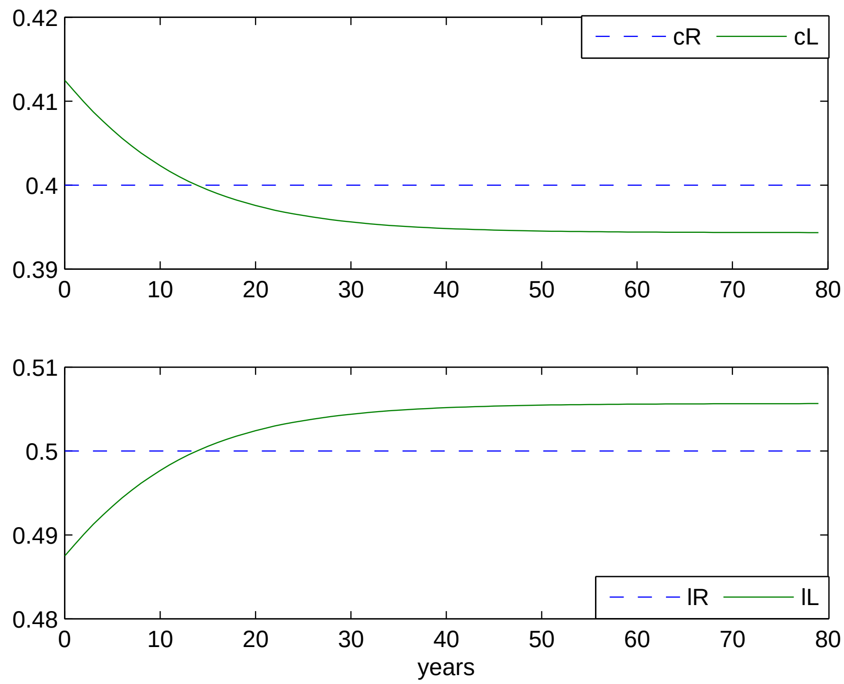

Figure 5: Consumption and leisure under RE and under learning dynamics

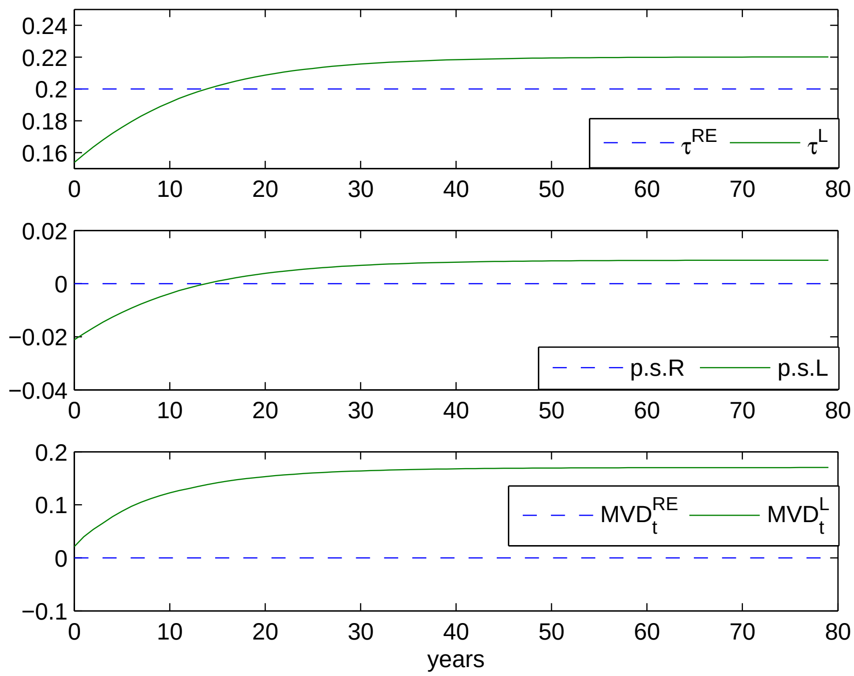

Figure 6: Taxes, primary surplus and debt under RE and under learning dynamics

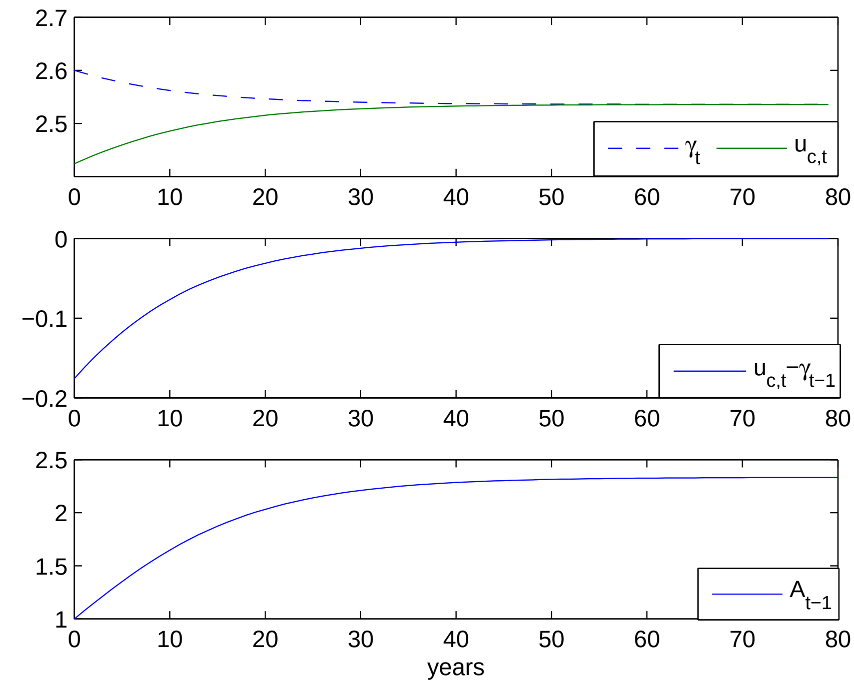

Figure 7: Forecast errors, history and non convergence to the RE values

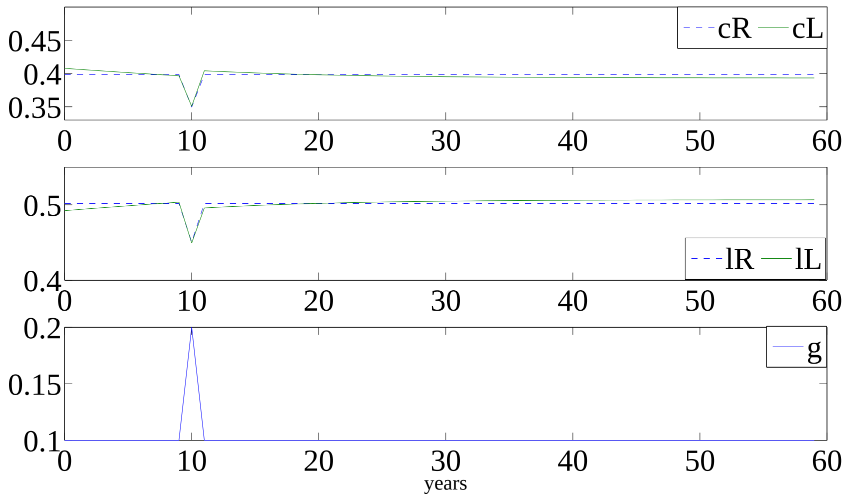

Figure 8: Consumption and leisure under RE and under learning dynamics

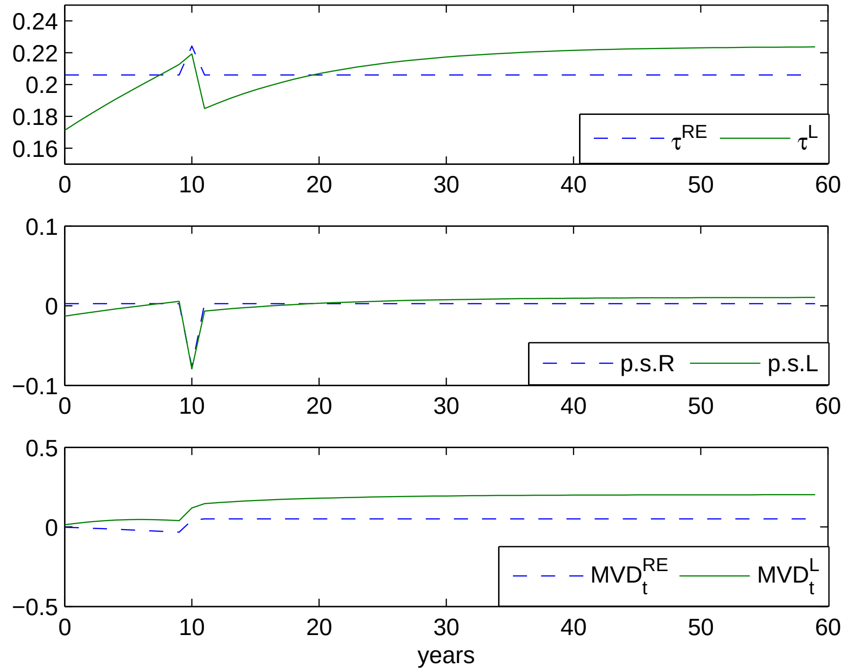

Figure 9: Taxes, primary surplus and debt under RE and under learning dynamics

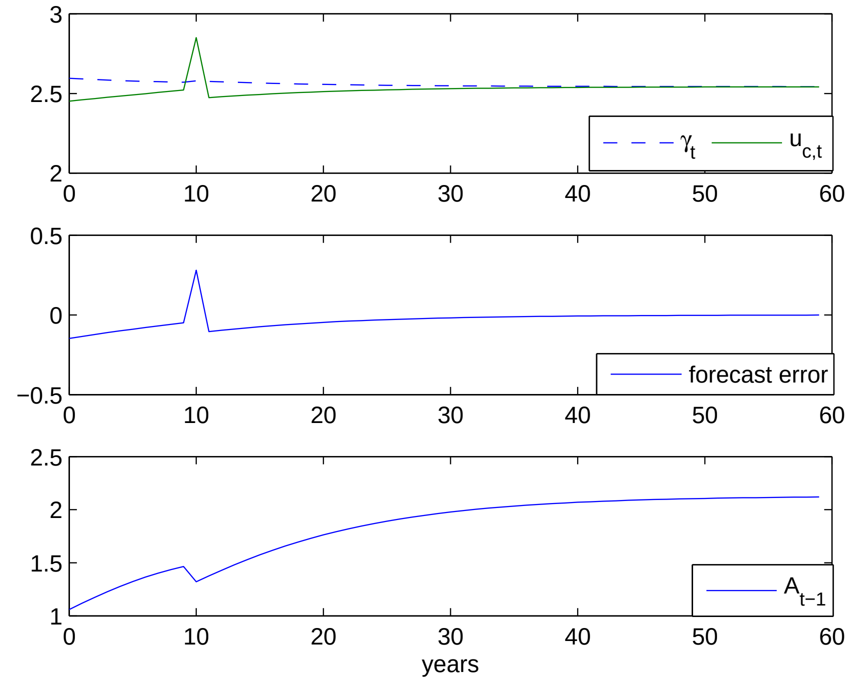

Figure 10: Forecast errors, history and non convergence to the RE values

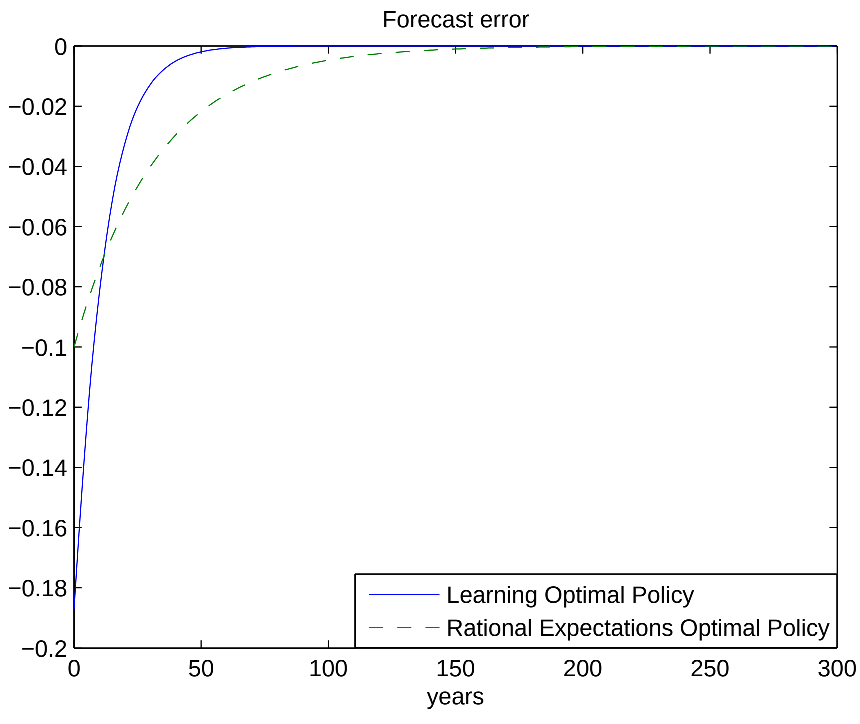

Figure 11: Forecast Error

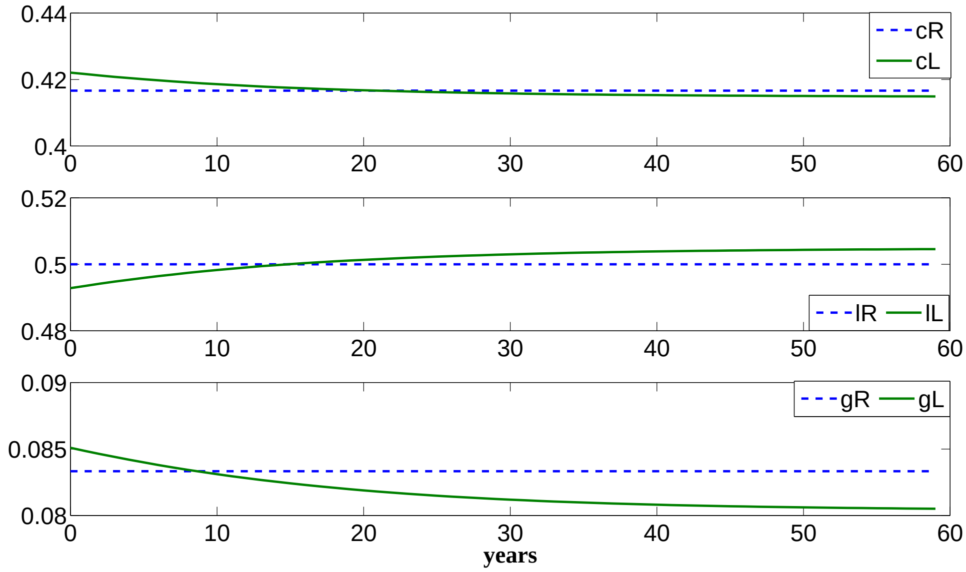

Figure 12: Consumption and leisure under RE and under learning dynamics

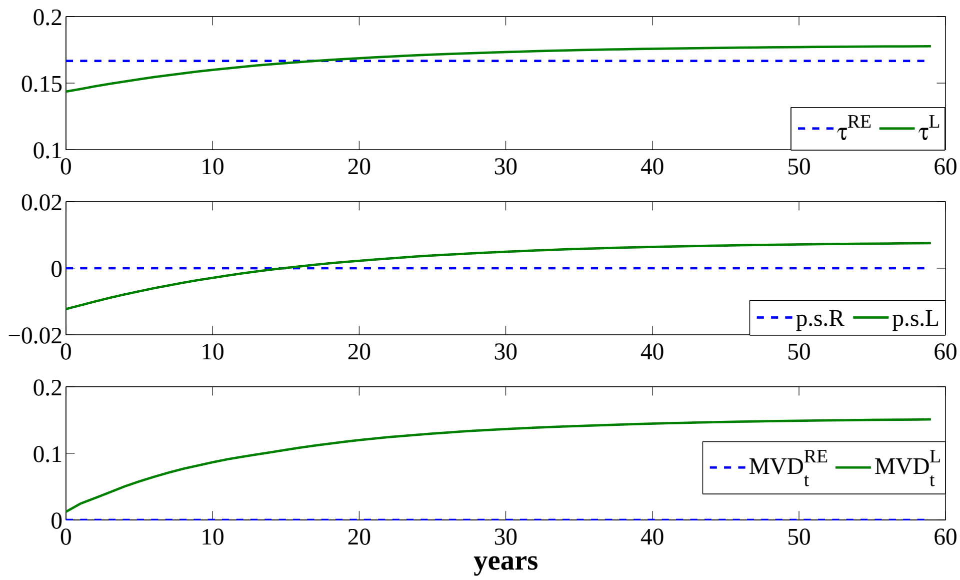

Figure 13: Taxes, primary surplus and debt under RE and under learning dynamics

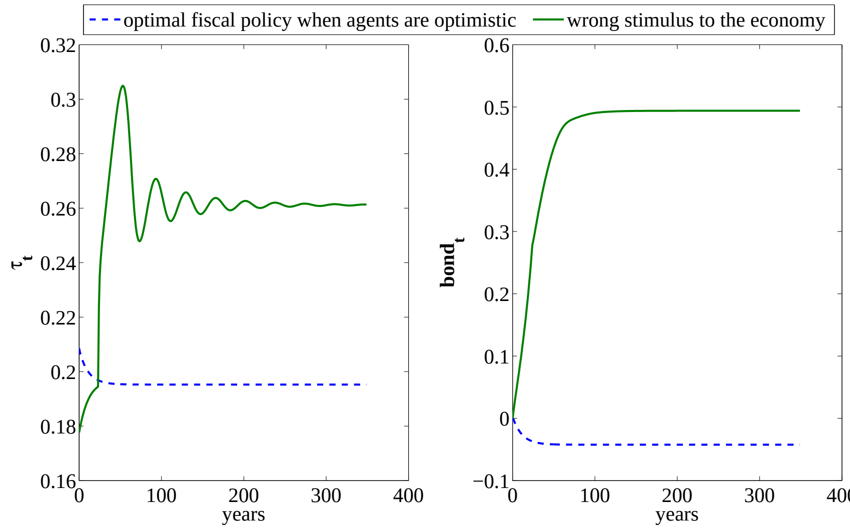

Figure 14:

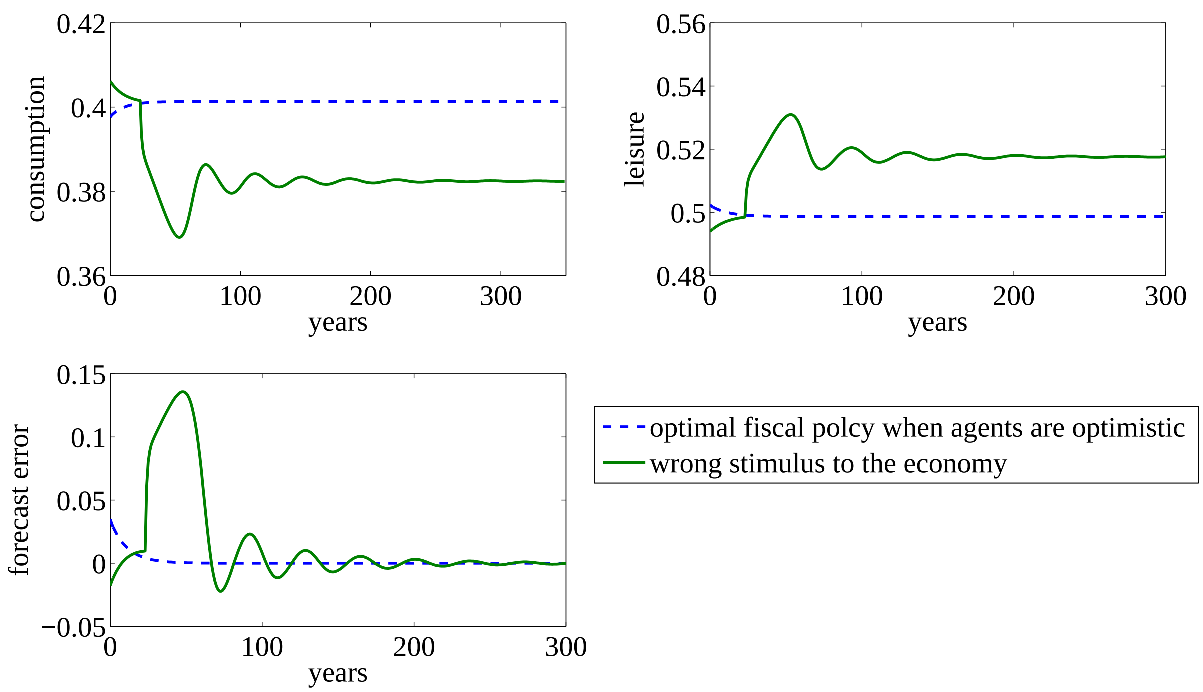

Figure 15:

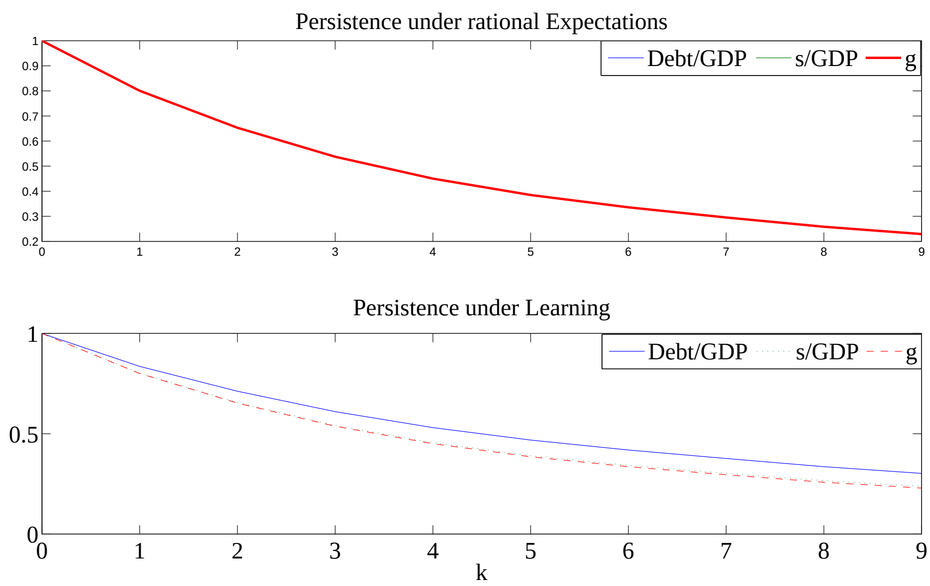

Figure 16: Top Panel: RE framework; Bottom Panel: Learning framework

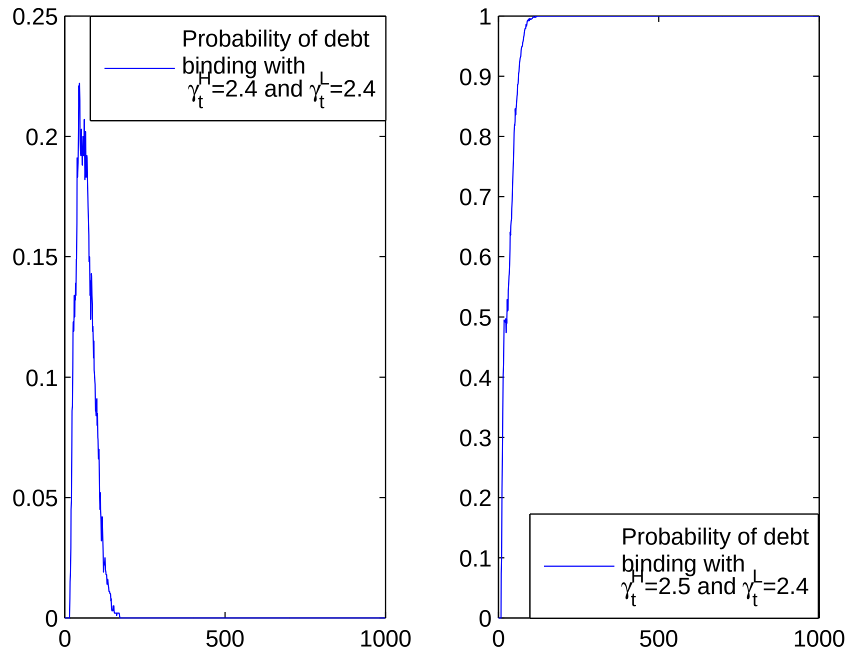

Figure 17: Comparison of probabilities of binding debt limits

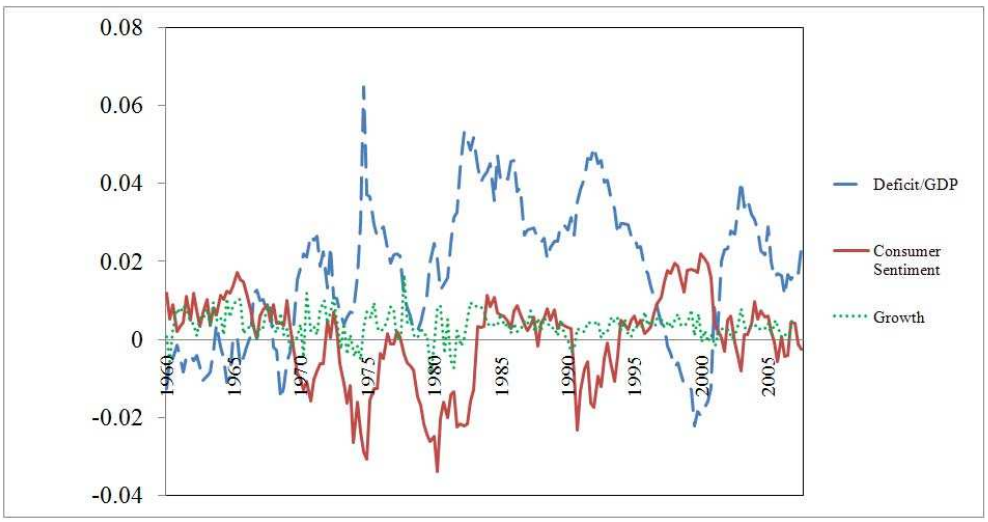
0.08
0.06
0.04
0.02
0
-0.02
-0.04
1960
1965
1970
1975
1980
1985
1990
1995
2000
2005
Deficit/GDP
Consumer Sentiment
Growth

Figure 18: